$\mathbb{Z}_2\mathbb{Z}_4$-Additive Codes

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Outline

1. Introduction
   - Codes over rings
   - Binary codes
   - Quaternary codes

Consider a principal ideal ring R.

A code C of length n is a subset of R^n. If C is a subgroup, then C is an additive code over R.

The **dual code** of C is defined in the standard way by

$$C^{\perp} = \{\mathbf{v} \in R^n \mid \mathbf{u} \cdot \mathbf{v} = 0, \text{ for all } \mathbf{u} \in C\},$$

where $\mathbf{u} \cdot \mathbf{v} = \sum_{i=0}^{n-1} u_i v_i \in R$.

What rings are we interested on?

1. Binary linear codes; $R = \mathbb{Z}_2$.
2. Quaternary linear codes; $R = \mathbb{Z}_4$.

   [HKC+94] A.R. Hammons, P.V. Kumar, A.R. Calderbank, N.J.A. Sloane, P. Solé.
   The $\mathbb{Z}_4$-linearity of kerdock, preparata, goethals and related codes.
   IEEE Trans. Info. Theory, vol. 40, pp. 301-319, 1994.

3. Codes having binary and quaternary coordinates!

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3. Codes having binary and quaternary coordinates!

Why only binary and quaternary coordinates?

A little bit of history...

1973 Additive codes were defined by Delsarte in terms of association schemes $(X, R = \{R_0, \dots, R_d\})$.

[Del73] P. Delsarte.
An algebraic approach to the association schemes of coding theory.
Philips Res. Rep. Suppl., vol. 10, pp. iv–97, 1973.

[DL98] P. Delsarte, V. I. Levenshtein.
Association schemes and coding theory,
IEEE Transactions on Information Theory, vol. 44, pp. 2477-2504, 1998.

An additive code is a subgroup of the underlying abelian group in a translation-invariant association scheme:

- X has abelian group structure,
- $(x, y) \in R_i \longrightarrow (x+z, y+z) \in R_i$, for $i \in \{1, \dots, d\}$, $x, y, z \in X$.

A little bit of history...

1997 Translation-invariant propelinear codes were defined by Rifà and Pujol.

[RP97] J. Rifà, J. Pujol.
Translation-invariant propelinear codes
IEEE Transactions on Information Theory, vol. 43, pp. 590-598, 1997.

$C \subseteq \mathbb{Z}_2^n$ is called a propelinear code if $\forall v \in C$ there exists $\pi_v \in S_n$ such that:

- i) $\forall c \in C : v + \pi_v(c) \in C$,
- ii) $\forall c \in C : \pi_v \circ \pi_c = \pi_m$, where $m = v + \pi_v(c)$.

These codes are group-ismorphic to subgroups of

$$\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Q}_8^\sigma,$$

where $\mathbb{Q}_8$ is the non-abelian quaternion group on 8 elements.

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From [RP97] and [DL98]...

...codes that are subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ are the only additive codes in the binary Hamming scheme.

1 Introduction
- Codes over rings
- Binary codes
- Quaternary codes

Binary codes

Let $C \subseteq \mathbb{Z}_2^n$ be a binary code.
If C is a subgroup of $\mathbb{Z}_2^n$, then C is a binary linear code.

Two binary codes C_1 and C_2 of length n are equivalent if there exists a vector $a \in \mathbb{Z}_2^n$ and a coordinate permutation $\pi \in S_n$ such that $C_2 = \{a + \pi(c) \mid c \in C_1\}$.

They are permutation-equivalent or isomorphic if there exists a coordinate permutation $\pi \in S_n$ such that $C_2 = \{\pi(c) \mid c \in C_1\}$.

Example 1.

Let C be a binary linear code of length 5 and dimension 2, with generator matrix

$$G = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

The dual code $C^\perp = \{v \in \mathbb{Z}_2^n \mid u \cdot v = 0 \text{ for all } u \in C\}$ is a binary linear code of length 5 and dimension 3, with generator matrix

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix H is a generator matrix of $C^\perp$ and a parity-check matrix of C.

The code C has 2^2 codewords and its dual code $C^\perp$ has 2^3 codewords, so $|C| \cdot |C^\perp| = 2^2 \cdot 2^3 = 2^5$.

Quaternary codes

A quaternary linear code $\mathcal{C}$ is a subgroup of $\mathbb{Z}_4^n$.

Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_4^n$, it is isomorphic to an abelian structure like $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$.

Its order is a power of two and its type is of the form $2^\gamma 4^\delta$.

- The number of codewords is $|\mathcal{C}| = 2^\gamma 4^\delta$.
- The number of order two codewords is $2^{\gamma+\delta}$.

Proposition 1 (HKC+94).

Any quaternary linear code $\mathcal{C}$ of length n and type $4^\delta 2^\gamma$ is permutation equivalent to a quaternary linear code with generator matrix of the form

$$\mathcal{G}_S = \begin{pmatrix} 2T & 2I_\gamma & \mathbf{0} \\ S & R & I_\delta \end{pmatrix}, \qquad (1)$$

where R, T are matrices over $\mathbb{Z}_2$ of size $\delta \times \gamma$ and $\gamma \times (n-\gamma-\delta)$, respectively; and S is a matrix over $\mathbb{Z}_4$ of size $\delta \times (n-\gamma-\delta)$.

Proposition 2 (HKC+94).

The quaternary dual code $\mathcal{C}^\perp$ of the quaternary linear code $\mathcal{C}$ of length n with generator matrix $\mathcal{G}_S$ as (1) has generator matrix

$$\mathcal{H}_S = \begin{pmatrix} \mathbf{0} & 2I_\gamma & 2R^t \\ I_{n-\gamma-\delta} & T^t & -(S+RT)^t \end{pmatrix}, \qquad (2)$$

where R, T are matrices over $\mathbb{Z}_2$ of size $\delta \times \gamma$ and $\gamma \times (n-\gamma-\delta)$, respectively; and S is a matrix over $\mathbb{Z}_4$ of size $\delta \times (n-\gamma-\delta)$.

Gray map. $\mathbb{Z}_4$-linear codes

The usual Gray map $\phi : \mathbb{Z}_4 \to \mathbb{Z}_2^2$ is defined as

$$\phi(0) = 00, \quad \phi(1) = 01, \quad \phi(2) = 11, \quad \phi(3) = 10.$$

Then, the (exended) Gray map is $\phi : \mathbb{Z}_4^n \to \mathbb{Z}_2^{2n}$

$$\phi(x_1, \dots, x_n) \to (\phi(x_1), \dots \phi(x_n)).$$

Quaternary linear codes can be viewed as binary codes under the usual Gray map. If $\mathcal{C}$ is a quaternary linear code, then the corresponding binary code $C = \phi(\mathcal{C})$ is said to be a $\mathbb{Z}_4$-linear code.

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Two quaternary codes $\mathcal{C}_1$ and $\mathcal{C}_2$ both of length n are monomially equivalent if one can be obtained from the other by permutating the coordinates and (if necessary) changing the signs of certain coordinates.

They are permutation equivalent if they differ only by a permutation of coordinates.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ both of length n are monomially equivalent, then $\phi(\mathcal{C}_1)$ and $\phi(\mathcal{C}_2)$ are permutation equivalent.

Bibliography

[BFR+10] J. Borges, C. Fernández-Córdoba, J. Rifà, J. Pujol and M. Villanueva.
$\mathbb{Z}_2\mathbb{Z}_4$-linear codes: generator matrices and duality
Designs, Codes and Cryptography, vol. 54, pp. 167-179, 2010.

Definitions

If $\mathcal{C}$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, then $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code.

For a vector $\mathbf{u} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, we write $\mathbf{u} = (u \mid u')$, where $u \in \mathbb{Z}_2^\alpha$ and $u' \in \mathbb{Z}_4^\beta$.

A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, so it is also isomorphic to an abelian structure like $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$.
Let $\mathcal{C}_b$ be the subcode of $\mathcal{C}$ which contains all codewords of order at most 2.

- The order of $\mathcal{C}$ is $|\mathcal{C}| = 2^\gamma 4^\delta$.
- The number codewords of order at most two in $\mathcal{C}$ is $|\mathcal{C}_b| = 2^{\gamma+\delta}$.

Example 2.

$$\mathcal{C}_1 = \{(00 \mid 0000), (11 \mid 2211), (00 \mid 0022), (11 \mid 2233)$$
$$(10 \mid 2020), (01 \mid 0231), (10 \mid 2002), (01 \mid 0213)\}$$

$$(\mathcal{C}_1)_b = \{(00 \mid 0000), (00 \mid 0022), (10 \mid 2020), (10 \mid 2002)\}$$

- $\mathcal{C}_1 \subseteq \mathbb{Z}_2^2 \times \mathbb{Z}_4^4$,
- $|\mathcal{C}_1| = 2^{\gamma+2\delta} = 8$,
- $|(\mathcal{C}_1)_b| = 2^{\gamma+\delta} = 4$.

$$\Longrightarrow \alpha = 2, \beta = 4, \gamma = 2, \delta = 2.$$

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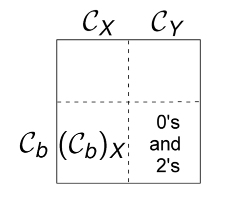
$$\Longrightarrow \alpha = 2, \beta = 4, \gamma = 2, \delta = 2.$$

Let X (respectively Y) be the set of $\mathbb{Z}_2$ (respectively $\mathbb{Z}_4$) coordinate positions, so $|X| = \alpha$ and $|Y| = \beta$. Unless otherwise stated, the set X corresponds to the first α coordinates and Y corresponds to the last β coordinates.

Call $\mathcal{C}_X$ (respectively $\mathcal{C}_Y$) the punctured code of $\mathcal{C}$ by deleting the coordinates out of X (respectively Y).

Let κ be the dimension of $(\mathcal{C}_b)_X$, which is a binary linear code. For the case $\alpha = 0$, we will write $\kappa = 0$.



Then, we will say that $\mathcal{C}$ is of type $(\alpha, \beta; \gamma, \delta; \kappa)$.

Example 3 (Cont. Example 1).

$$\begin{aligned}\mathcal{C}_1 = \{&(00 \mid 0000), (11 \mid 2211), (00 \mid 0022), (11 \mid 2233)\\ &(10 \mid 2020), (01 \mid 0231), (10 \mid 2002), (01 \mid 0213)\}\end{aligned}$$

$$(\mathcal{C}_1)_b = \{(00 \mid 0000), (00 \mid 0022), (10 \mid 2020), (10 \mid 2002)\}$$

$$((\mathcal{C}_1)_b)_X = \{(00), (10)\}$$

$\mathcal{C}_1$ is of type $(2, 4; 2, 2; 1)$

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The $\mathbb{Z}_2\mathbb{Z}_4$-additive codes of type $(\alpha, \beta; \gamma, \delta; \kappa)$ are a generalization of binary linear codes and quaternary linear codes.

- If $\beta = 0$, the $\mathbb{Z}_2\mathbb{Z}_4$-additive code is a binary linear code.

  In general, any binary linear code of length n and dimension k, an $[n, k]$ code, is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(n, 0; k, 0; k)$.

- If $\alpha = 0$, the $\mathbb{Z}_2\mathbb{Z}_4$-additive code is a quaternary linear code.

  In general, any quaternary linear code of length n and type $2^\gamma 4^\delta$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(0, n; \gamma, \delta; 0)$.

Counting $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

Theorem 4 (DS15).

The number of distinct $\mathbb{Z}_2\mathbb{Z}_4$-additive codes of type $(\alpha, \beta; \gamma, \delta; \kappa)$ is

$$2^{(\alpha+\beta-\gamma-\delta)\delta+(\beta-\delta-\gamma+\kappa)\kappa} \begin{bmatrix} \beta \\ \delta \end{bmatrix}_2 \begin{bmatrix} \alpha \\ \kappa \end{bmatrix}_2 \begin{bmatrix} \beta-\delta \\ \gamma-\kappa \end{bmatrix}_2,$$

where $\begin{bmatrix} x \\ k \end{bmatrix}_2$ *is the binary Gaussian binomial coefficient for* $k \geq 0$ *and* x *a real number.*

[DS15] S.T. Dougherty, E. Salturk.
Counting $\mathbb{Z}_2\mathbb{Z}_4$-Additive Codes.
NoncommutativeRings and Their Applications, Contemporary Mathematics, vol. 634, pp. 137-147, 2015.

Separable codes

A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is said to be separable if $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$.

Example 5.

Let $\mathcal{C}$ be the code

$$\mathcal{C} = \{(00 \mid 00), (00 \mid 12), (00 \mid 20), (00 \mid 32)\\ (11 \mid 00), (11 \mid 12), (11 \mid 20), (11 \mid 32)\}.$$

We have

$$\mathcal{C}_X = \{00, 11\},\\ \mathcal{C}_Y = \{00, 12, 20, 32\}.$$

Then, $\mathcal{C}$ is separable: $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$.

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$$\mathcal{C}_1 = \{(00 \mid 0000), (11 \mid 2211), (00 \mid 0022), (11 \mid 2233) \\ (10 \mid 2020), (01 \mid 0231), (10 \mid 2002), (01 \mid 0213)\}$$

We have

$$(\mathcal{C}_1)_X = \{00, 10, 01, 11\},$$
$$(\mathcal{C}_1)_Y = \{0000, 2211, 0022, 2233, \\ 2020, 0231, 2002, 0213\}.$$

Then, $\mathcal{C}_1$ is not separable: $\mathcal{C}_1 \neq (\mathcal{C}_1)_X \times (\mathcal{C}_1)_Y$.

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$$\begin{aligned}\mathcal{C}_1 = \{&(00 \mid 0000), (11 \mid 2211), (00 \mid 0022), (11 \mid 2233)\\ &(10 \mid 2020), (01 \mid 0231), (10 \mid 2002), (01 \mid 0213)\}\end{aligned}$$

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...some more parameters

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Let $\kappa_1 \leq \kappa$ and $\delta_2 \leq \delta$ such that

1. $\{(u \mid \mathbf{0}) \in \mathcal{C}\}$ is of type $(\alpha, \beta; \kappa_1, 0; \kappa_1)$,
2. $\langle\{(\mathbf{0} \mid u') \in \mathcal{C} : u' = \mathbf{0}$ or the order of u' is four$\}\rangle$ is of type $(\alpha, \beta; \gamma', \delta_2; 0)$ for an integer $\gamma' \leq \gamma$.

Consider the values κ_2 and δ_1 such that

$$\kappa = \kappa_1 + \kappa_2 \quad \text{and} \quad \delta = \delta_1 + \delta_2. \tag{3}$$

1. $\mathcal{C}_X$ is a binary linear $[\alpha, \kappa + \delta_1]$ code.
2. $\mathcal{C}_Y$ is a quaternary linear code of length β and type $2^{\gamma-\kappa_1}4^{\delta}$.
3. $\mathcal{C}$ is separable if and only if κ_2 and δ_1 are zero; that is, $\kappa = \kappa_1$ and $\delta = \delta_2$.

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Example 7.

Let $\mathcal{C}$ be the code of type $(2,2;1,1;1)$

$$\mathcal{C} = \{(00 \mid 00),(00 \mid 12),(00 \mid 20),(00 \mid 32)\\ (11 \mid 00),(11 \mid 12),(11 \mid 20),(11 \mid 32)\}.$$

- $\{(u \mid \mathbf{0}) \in \mathcal{C}\} = \{(00 \mid 00),(11 \mid 00)\}$ is of type $(2,2;\mathbf{1},0;\mathbf{1})$; $\kappa_1 = 1, \kappa_2 = 0$.
- $\langle\{(\mathbf{0} \mid u') \in \mathcal{C} : u' = \mathbf{0}$ or the order of u' is four$\}\rangle = \langle\{(00 \mid 00),(00 \mid 12)\}\rangle$ is of type $(2,2;0,\mathbf{1};0)$; $\delta_2 = 1, \delta_1 = 0$.
- $\mathcal{C}_X = \{00, 11\}$ is a linear $[2, 1+0]$ code.
- $\mathcal{C}_Y = \{00, 12, 20, 32\}$ is a quaternary linear code of lenght 2 and type $2^{1-1}4^1$.
- Since $\kappa = \kappa_1$ and $\delta = \delta_2$, $\mathcal{C}$ is separable.

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Example 8.

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- $\{(u \mid \mathbf{0}) \in \mathcal{C}\} = \{(00 \mid 0000)\}$ is of type $(2,4;\mathbf{0},0;\mathbf{0})$; $\kappa_1 = 0, \kappa_2 = 1$.
- $\langle\{(\mathbf{0} \mid u') \in \mathcal{C} : u' = \mathbf{0}$ or the order of u' is four$\}\rangle = \langle\{(00 \mid 0000)\}\rangle$ is of type $(2,4;0,\mathbf{0};0)$; $\delta_2 = 0, \delta_1 = 1$.
- $(\mathcal{C}_1)_X = \{00, 10, 01, 11\}$ is a linear $[2, 1+1]$ code.
- $(\mathcal{C}_1)_Y = \{0000, 2211, 0022, 2233, 2020, 0231, 2002, 0213\}$ is a quaternary linear code of lenght 2 and type $2^{1-0}4^1$.
- Since $\kappa \neq \kappa_1$ (or $\delta \neq \delta_2$), $\mathcal{C}_1$ is not separable.

Example 8.

Let $\mathcal{C}_1$ be the code of type $(2,4;1,1;1)$

$$\mathcal{C}_1 = \{(00 \mid 0000), (11 \mid 2211), (00 \mid 0022), (11 \mid 2233)$$
$$(10 \mid 2020), (01 \mid 0231), (10 \mid 2002), (01 \mid 0213)\}$$

- $\{(u \mid \mathbf{0}) \in \mathcal{C}\} = \{(00 \mid 0000)\}$ is of type $(2,4;\mathbf{0},0;\mathbf{0})$; $\kappa_1 = 0, \kappa_2 = 1$.
- $\langle\{(\mathbf{0} \mid u') \in \mathcal{C} : u' = \mathbf{0}$ or the order of u' is four$\}\rangle = \langle\{(00 \mid 0000)\}\rangle$ is of type $(2,4;0,\mathbf{0};0)$; $\delta_2 = 0, \delta_1 = 1$.
- $(\mathcal{C}_1)_X = \{00, 10, 01, 11\}$ is a linear $[2, 1+1]$ code.
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- $\{(u \mid \mathbf{0}) \in \mathcal{C}\} = \{(00 \mid 0000)\}$ is of type $(2,4;\mathbf{0},0;\mathbf{0})$; $\kappa_1 = 0, \kappa_2 = 1$.
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- $(\mathcal{C}_1)_X = \{00, 10, 01, 11\}$ is a linear $[2, 1+1]$ code.
- $(\mathcal{C}_1)_Y = \{0000, 2211, 0022, 2233, 2020, 0231, 2002, 0213\}$ is a quaternary linear code of lenght 2 and type $2^{1-0}4^1$.
- Since $\kappa \neq \kappa_1$ (or $\delta \neq \delta_2$), $\mathcal{C}_1$ is not separable.

Two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are monomially equivalent if one can be obtained from the other by permutating the coordinates and (if necessary) changing the signs of certain coordinates over $\mathbb{Z}_4$.

They are permutation equivalent if they differ only by a permutation of coordinates.

Gray map. $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

The Gray map is $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \rightarrow \mathbb{Z}_2^{\alpha+2\beta}$:

$$\Phi(x_1, \ldots, x_\alpha, x_{\alpha+1}, \ldots, x_{\alpha+\beta}) \rightarrow (x_1, \ldots, x_\alpha, \phi(x_{\alpha+1}), \ldots \phi(x_{\alpha+\beta})).$$

As for quaternary linear codes, $\mathbb{Z}_2\mathbb{Z}_4$-additive codes can be view as binary codes under the Gray map.

If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, then the corresponding binary code $C = \Phi(\mathcal{C})$ is said to be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of length $n = \alpha + 2\beta$ and type $(\alpha, \beta; \gamma, \delta; \kappa)$, where γ, δ and κ are defined as above.

If two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are monomially equivalent, then, after the Gray map, the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes $C_1 = \Phi(\mathcal{C}_1)$ and $C_2 = \Phi(\mathcal{C}_2)$ are permutation equivalent as binary codes.

Note that the inverse statement is not always true.

Generator matrices

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. Although $\mathcal{C}$ is not a free module, every codeword is uniquely expressible in the form

$$\sum_{i=1}^{\gamma} \lambda_i u_i + \sum_{j=1}^{\delta} \mu_j v_j,$$

where $\lambda_i \in \mathbb{Z}_2$ for $1 \leq i \leq \gamma$, $\mu_j \in \mathbb{Z}_4$ for $1 \leq j \leq \delta$ and $u_i, v_j \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ of order two and order four, respectively.

The vectors $\{u_i\}_{i=1}^{\gamma}, \{v_j\}_{j=1}^{\delta}$ give us a generator matrix $\mathcal{G}$ of $\mathcal{C}$ of size $(\gamma+\delta)\times(\alpha+\beta)$ and of the form

$$\mathcal{G} = \left(\begin{array}{c|c} B_1 & 2B_3 \\ B_2 & Q \end{array}\right),$$

where B_1, B_2 are matrices over $\mathbb{Z}_2$ of size $\gamma\times\alpha$ and $\delta\times\alpha$, resp.; and B_3, Q are matrices over $\mathbb{Z}_4$ of size $\gamma\times\beta$ and $\delta\times\beta$, resp. In B_3 all entries are in $\{0,1\}$ and in Q all row vector is of order four.

Theorem 3 (BFR+10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $\mathcal{C}$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix in standard the form

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_\kappa & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S_b & S_q & R & I_\delta \end{array} \right), \quad (4)$$

where T_b, S_b are matrices over $\mathbb{Z}_2$ and S_q, T_1, T_2, R is a matrix over $\mathbb{Z}_4$, and all the entries of T_1, T_2 and R are in $\{0, 1\}$.

Lemma 4 (BFR+10).

There exists a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ of type $(\alpha, \beta; \gamma, \delta; \kappa)$ if and only if

$$\begin{array}{c} \alpha, \beta, \gamma, \delta, \kappa \geq 0, \quad \alpha + \beta > 0, \\ 0 < \delta + \gamma \leq \beta + \kappa \quad \textit{and} \quad \kappa \leq \min(\alpha, \gamma). \end{array} \quad (5)$$

Example 9.

Let $\mathcal{C}_1$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(2,4;1,1;1)$

$$\mathcal{C}_1 = \{(00 \mid 0000), (11 \mid 2211), (00 \mid 0022), (11 \mid 2233)$$
$$(10 \mid 2020), (01 \mid 0231), (10 \mid 2002), (01 \mid 0213)\}.$$

We have that $\mathcal{C}_1$ is generated by

$$\mathcal{G}_1 = \left(\begin{array}{cc|cccc} \mathbf{1} & 0 & 2 & 0 & 2 & 0 \\ 1 & 1 & 2 & 2 & 1 & \mathbf{1} \end{array}\right).$$

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix in standard form

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_\kappa & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S_b & S_q & R & I_\delta \end{array}\right),$$

Then, $\mathcal{C}$ is permutation equivalent to a code with generator matrix as

$$\mathcal{G}' = \left(\begin{array}{cccc|ccccc} I_{\kappa_1} & T_{b_1} & T_{b_2} & T_{b_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{\kappa_2} & T_{b_4} & T_{b_5} & 2T_2 & 2T_2' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2T_1 & 2T_1' & 2I_{\gamma-\kappa} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_{b_1} & S_{b_2} & S_{q_1} & S_{q_2} & R_1 & I_{\delta_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & S_{q_3} & S_{q_4} & R_2 & R_3 & I_{\delta_2} \end{array}\right), \qquad (6)$$

where T_{b_i}, S_{b_j} are matrices over $\mathbb{Z}_2$, S_{q_k}, R_s, T_t are quaternary matrices, and all the entries of T_t are in $\{0, 1\}$.

Example 10.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix in standard form $\mathcal{G}_S$

$$\mathcal{G}_S = \left(\begin{array}{c|c} 1111 & 000000 \\ 0101 & 220000 \\ 0000 & 202000 \\ 0101 & 000200 \\ 0101 & 111010 \\ 0011 & 101101 \end{array}\right); \mathcal{G}' = \left(\begin{array}{c|c} 1111 & 000000 \\ 0101 & 220000 \\ 0000 & 202000 \\ 0101 & 000200 \\ 0011 & 101110 \\ 0000 & 111201 \end{array}\right)$$

$\mathcal{C}$ is permutation equivalent to a code generated by $\mathcal{G}'$. Therefore, $\kappa_1 = 1$ and $\delta_2 = 1$.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive separable code; $\kappa = \kappa_1$, $\delta = \delta_2$

$$\mathcal{G} = \left(\begin{array}{cccc|ccccc} I_{\kappa_1} & T_{b_1} & T_{b_2} & T_{b_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \cancel{\mathbf{0}} & \cancel{I_{\kappa_2}} & \cancel{T_{b_4}} & \cancel{T_{b_5}} & \cancel{2T_2} & \cancel{2T_2'} & \cancel{\mathbf{0}} & \cancel{\mathbf{0}} & \cancel{\mathbf{0}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2T_1 & 2T_1' & 2I_{\gamma-\kappa} & \mathbf{0} & \mathbf{0} \\ \cancel{\mathbf{0}} & \cancel{\mathbf{0}} & \cancel{S_{b_1}} & \cancel{S_{b_2}} & \cancel{S_{q_1}} & \cancel{S_{q_2}} & \cancel{R_1} & \cancel{I_{\delta_1}} & \cancel{\mathbf{0}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & S_{q_3} & S_{q_4} & R_2 & R_3 & I_{\delta_2} \end{array}\right)$$

$$\Downarrow$$

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_\kappa & T_b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_q & R & I_\delta \end{array}\right),$$

2 $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

- Definitions
- Generator matrices
- Dual codes. Parity-check matrices
- Coding and decoding

Duality of codes over rings.

Let R be a principal ideal ring.
The inner product for any two vectors $u, v \in R^n$ is defined as:

$$u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n \in R.$$

Let $\mathcal{C} \subseteq R^n$ be a linear code of length n over R. The dual code of $\mathcal{C}$, denoted by $\mathcal{C}^\perp$, is defined in the standard way:

$$\mathcal{C}^\perp = \{v \in R^n \mid u \cdot v = 0 \text{ for all } u \in \mathcal{C}\}.$$

It is easy to see that $\mathcal{C}^\perp$ is a subgroup of R^n, so $\mathcal{C}^\perp$ is also a quaternary linear code.

Dual of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. Parity-check matrices

What if we have a code $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$????

Fundamental theorem of finite Abelian groups

The fundamental theorem of finite Abelian groups states that a finite Abelian group G is isomorphic to

$$\langle p_1^{\alpha_1} \rangle \times \cdots \times \langle p_k^{\alpha_k} \rangle,$$

where $p_1, \ldots, p_k$ are not necessarily distinct prime numbers, and $\alpha_i \geq 1$ for any $i \in \{1, \ldots, k\}$.

- The decomposition is unique up to the order in which the factors are written.
- $\{p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}\}$ is a basis.
- The exponent of G is $m = \operatorname{lcm}\{{p_i}^{\alpha^i} \mid i = 1, \ldots, k\}$.

Fundamental theorem of finite Abelian groups

For $i \in \{i, \ldots, k\}$, select s_i such that $m = s_i {p_i}^{\alpha_i}$ (s_i is the order of ${p_i}^{\alpha_i}$ in $\mathbb{Z}_m$).

The inner product of elements $u = (u_1, u_2, \ldots, u_k)$ and $v = (v_1, v_2, \ldots, v_k) \in G$ is uniquely defined as the equivalence class of

$$\sum_{i=1}^{k} s_i u_i v_i \in \mathbb{Z}_m.$$

Fundamental theorem of finite Abelian groups: $G = \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$

$$G = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \cdots \times \mathbb{Z}_4.$$

1. The exponent of G is $m = 4$.
2. $m = s_i \cdot 2$, for $i \in \{1, \ldots, \alpha\} \Rightarrow s_i = 2$,
3. $m = s_j \cdot 4$, for $j \in \{\alpha+1, \ldots, \alpha+\beta\} \Rightarrow s_j \in \{1, 3\}$.

For $u = (u_1, u_2, \ldots, u_{\alpha+\beta})$ and $v = (v_1, v_2, \ldots, v_{\alpha+\beta}) \in G$,

$$u \cdot v = \sum_{i=1}^{\alpha+\beta} s_i u_i v_i = \sum_{i=1}^{\alpha} 2u_i v_i + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j \in \mathbb{Z}_4.$$

Dual of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. Parity-check matrices

The inner product for any two vectors $u, v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is defined as:

$$u \cdot v = 2(\sum_{i=1}^{\alpha} u_i v_i) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j \in \mathbb{Z}_4.$$

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. The additive dual code of $\mathcal{C}$, denoted by $\mathcal{C}^\perp$, is defined in the standard way:

$$\mathcal{C}^\perp = \{v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \mid u \cdot v = 0 \text{ for all } u \in \mathcal{C}\}.$$

It is easy to see that $\mathcal{C}^\perp$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, so $\mathcal{C}^\perp$ is also a $\mathbb{Z}_2\mathbb{Z}_4$-additive code.

- If $\mathcal{C} \subset \mathcal{C}^\perp$, $\mathcal{C}$ is called an additive self-orthogonal code.
- If $\mathcal{C} = \mathcal{C}^\perp$, $\mathcal{C}$ is called an additive self-dual code.

One could think on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes only as quaternary linear codes, changing ones by twos in the coordinates over $\mathbb{Z}_2$.

Example 11.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by

$$\mathcal{G} = \left(\begin{array}{cc|cccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 1 & 1 & 2 & 2 & 1 & 1 \end{array} \right).$$

$$\mathcal{C} = \{(00 \mid 0000), (11 \mid 2211), (00 \mid 0022), (11 \mid 2233)$$
$$(10 \mid 2020), (01 \mid 0231), (10 \mid 2002), (01 \mid 0213)\}$$

The code $\mathcal{C}$ can be seen as the quaternary linear code generated by

$$\left(\begin{array}{cccccc} 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 2 & 2 & 2 & 1 & 1 \end{array} \right).$$

$$\mathcal{C} = \{(000000), (222211), (000022), (222233)$$
$$(202020), (020231), (202002), (020213)\}$$

However...

...these quaternary linear codes are not equivalent to the quaternary linear codes!!

Note that the inner product defined in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ gives us that the dual code of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code is not equivalent to the dual code of the corresponding quaternary linear code.

Example 12.

Taking $\alpha = \beta = 1$ and the vectors $\mathbf{v} = (1 \mid 3)$ and $\mathbf{w} = (1 \mid 2)$, it is easy to check that $\mathbf{v} \cdot \mathbf{w} = 0$, so $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

Taking $\beta = 2$ and changing the ones by twos in the coordinates over $\mathbb{Z}_2$ of these vectors, we get $\bar{v} = (23)$ and $\bar{w} = (22)$, which are not orthogonal in the quaternary sense.

Example 13 (cont.).

- Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by

$$\left(\begin{array}{c|c} 1 & 3 \end{array} \right).$$

  Then, $\mathcal{C} = \{(0 \mid 0), (1 \mid 3), (0 \mid 2), (1 \mid 1)\}$ and $\mathcal{C}^\perp = \{(0 \mid 0), (1 \mid 2)\}$. Note that $\mathcal{C}$ is of type $(1,1;0,1;0)$ and $\mathcal{C}^\perp$ is of type $(1,1;1,0;1)$.

- The corresponding quaternary linear code $\mathcal{D}$ is generated by

$$\left(\begin{array}{|cc} 2 & 3 \end{array} \right).$$

  Then, $\mathcal{D} = \{(00), (23), (02), (21)\}$ and $\mathcal{D}^\perp = \{(00), (32), (20), (12)\}$. Note that $\mathcal{D}$ is of type $(0,2;0,1;0)$ and $\mathcal{D}^\perp$ is of type $(0,2;0,1;0)$.

Proposition 5 (HKC+94).

The quaternary dual code $\mathcal{C}^\perp$ of the quaternary linear code $\mathcal{C}$ of length n with generator matrix

$$\mathcal{G}_S = \begin{pmatrix} 2T & 2I_\gamma & \mathbf{0} \\ S & R & I_\delta \end{pmatrix}, \tag{7}$$

has generator matrix

$$\mathcal{H}_S = \begin{pmatrix} \mathbf{0} & 2I_\gamma & 2R^t \\ I_{n-\gamma-\delta} & T^t & -(S+RT)^t \end{pmatrix}, \tag{8}$$

where R, T, S are matrices over $\mathbb{Z}_4$ of size $\delta \times \gamma$, $\gamma \times (n-\gamma-\delta)$, and $\delta \times (n-\gamma-\delta)$ respectively; and all the entries in R and T are in $\{0,1\}$.

In order to construct the additive dual code of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, we will need the following maps:

- The usual one modulo two, $\xi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$, that is $\xi(0) = 0$, $\xi(1) = 1$, $\xi(2) = 0$, $\xi(3) = 1$.
- The identity map, $\iota : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$, that is $\iota(0) = 0$, $\iota(1) = 1$.
- The normal inclusion from the additive structure in $\mathbb{Z}_2$ to $\mathbb{Z}_4$, $\chi : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$, that is $\chi(0) = 0$, $\chi(1) = 2$.

These maps can be extended to the maps:

- $(\xi, Id) : \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ denoted also by ξ.
- $(\iota, Id) : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta$ denoted also by ι.
- $(\chi, Id) : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta$ denoted also by χ.

Let $(u \cdot v)_4$ denote the standard inner product for quaternary vectors u, v and $\mathbf{u} \cdot \mathbf{v}$ the inner product for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$.

Lemma 6 (BFR+10).

If $\mathbf{u} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, $v \in \mathbb{Z}_4^{\alpha+\beta}$, *then* $(\chi(\mathbf{u}) \cdot v)_4 = \mathbf{u} \cdot \xi(v)$.

Lemma 7 (BFR+10).

If $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, *then* $(\chi(\mathbf{u}) \cdot \iota(\mathbf{v}))_4 = \mathbf{u} \cdot \mathbf{v}$.

Proposition 8 (BFR+10).

Let $\mathcal{C}$ *be a* $\mathbb{Z}_2\mathbb{Z}_4$*-additive code of type* $(\alpha, \beta; \gamma, \delta; \kappa)$. *Then,*

$$\mathcal{C}^\perp = \xi(\chi(\mathcal{C})^\perp) \quad \textit{and} \quad \mathcal{C}^\perp = \chi^{-1}(\xi^{-1}(\mathcal{C})^\perp).$$

Theorem 9 (BFR+10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix in standard form

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_\kappa & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S_b & S_q & R & I_\delta \end{array}\right). \quad (9)$$

Then, the generator matrix of $\mathcal{C}^\perp$ is

$$\mathcal{H}_S = \left(\begin{array}{cc|ccc} T_b^t & I_{\alpha-\kappa} & \mathbf{0} & \mathbf{0} & 2\iota(S_b)^t \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2I_{\gamma-\kappa} & 2R^t \\ \xi(T_2)^t & \mathbf{0} & I_{\beta+\kappa-\gamma-\delta} & T_1^t & -\left(S_q + RT_1\right)^t \end{array}\right), \quad (10)$$

where T_b, S_b are matrices over $\mathbb{Z}_2$ and T_1, T_2, R, S_q are matrices over $\mathbb{Z}_4$ and all the entries in T_1 and T_2 are in $\{0, 1\}$.

Theorem 10 (BFR+10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. The additive dual code $\mathcal{C}^\perp$ is then of type $(\alpha, \beta; \bar{\gamma}, \bar{\delta}; \bar{\kappa})$, where

$$\begin{aligned} \bar{\gamma} &= \alpha + \gamma - 2\kappa, \\ \bar{\delta} &= \beta - \gamma - \delta + \kappa, \\ \bar{\kappa} &= \alpha - \kappa. \end{aligned} \tag{11}$$

Corollary 11 (BFR+10).

If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, then $|\mathcal{C}| \cdot |\mathcal{C}^\perp| = 2^\alpha 4^\beta$.

Example 14.

Let $\mathcal{C}_1$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(2,4;2,1;1)$ with generator matrix $\mathcal{G}_1$. The additive dual code $\mathcal{C}_1^\perp$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix $\mathcal{H}_1$.

$$\mathcal{G}_1 = \left(\begin{array}{cc|cccc} 1 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 3 & 1 & 1 & 1 \end{array}\right) \quad \mathcal{H}_1 = \left(\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 3 \end{array}\right)$$

- $\mathcal{H}_1$ is a generator matrix of $\mathcal{C}_1^\perp$ and a parity-check matrix of $\mathcal{C}_1$.
- The code $\mathcal{C}_1$ is of type $(2,4;2,1;1)$ and $\mathcal{C}_1^\perp$ is of type $(2,4;2,2;1)$.
- The code $\mathcal{C}_1$ has $2^2 4 = 2^4$ codewords and $\mathcal{C}_1^\perp$ has $2^2 4^2 = 2^6$ codewords, so $|\mathcal{C}_1| \cdot |\mathcal{C}_1^\perp| = 2^2 4^4 = 2^{10}$.

Again, in general the $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$ is not linear, so it need not have a dual. However, the corresponding binary code $C_\perp = \Phi(\mathcal{C}^\perp)$ is called $\mathbb{Z}_2\mathbb{Z}_4$-dual code of C.

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Phi} & C = \Phi(\mathcal{C}) \\
{\scriptstyle\perp}\big\downarrow & & \\
\mathcal{C}^\perp & \xrightarrow{\Phi} & C_\perp = \Phi(\mathcal{C}^\perp)
\end{array}
$$

- If $C \subset C_\perp$, C is called a self $\mathbb{Z}_2\mathbb{Z}_4$-orthogonal code.
- If $C = C_\perp$, C is called a self $\mathbb{Z}_2\mathbb{Z}_4$-dual code.

Example 15.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(1,2;0,2;0)$

- $\mathcal{C} = \{(0|000),(0|323),(1|330),(1|231),(1|132),(1|033),(0|220),(1|312),(0|121),(0|022),(1|213),(0|301),(0|202),(1|110),(0|103),(1|011)\}$.

We have that

- $\mathcal{C}^{\perp}=\{(0|000),(1|020),(1|111),(1|202),(0|131),(0|222),(0|313),(1|333)\}$,
- $C = \Phi(\mathcal{C}) = \{(0000000),(1000101),(0010010),(1010100),(0110011),(1110110),(0100001),(1100111),(0001111),(1001010),(0011101),(1011011),(0111100),(1111001),(0101110),(1101000)\}$ is a binary non-linear code, and
- $C_{\perp} = \Phi(\mathcal{C}^{\perp}) = \{(0000000),(1001100),(0011001),(1010101),(0111111),(1110011),(0100110),(1101010)\}$ is a binary linear code.

Binary coding. Example.

Let C be a binary Hamming code (linear 1-perfect code) of length 7 and dimension 4, that is, a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(7,0;4,0;4)$ generated by

$$\mathcal{G}_S = \left(\begin{array}{ccccccc|} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right).$$

- Information: $1011\ 1110... \rightarrow i_1 = (1011),\ i_2 = (1110)...$
- Encoding: $v_j = i_j \cdot \mathcal{G}_S$.
- Encoded info.: $v_1 = 1011010,\ v_2 = 1110000... \rightarrow 1011010\ 1110000\ ...$

Binary decoding: syndrome table.

Let C be an $[n,k,d]$ code with parity check matrix H with error correcting capability t. Consider $\{e_i\}_{i=1}^r$ all error vectors with $w_t(e_i) \leq t$.

error $\subseteq \mathbb{Z}_2^n$	syndrome $\subseteq \mathbb{Z}_2^{n-k}$
$\mathbf{0}$	$\mathbf{0}$
e_1	$s_1 = e_1 \cdot H^t$
$\vdots$	$\vdots$
e_r	$s_r = e_r \cdot H^t$

- For a received w, compute $s = w \cdot H^t$.
- If $s = s_j$, then decode by $v' = w - e_j$.

Binary decoding. Example of a perfect code.

The parity-check matrix of C is

$$\mathcal{H}_S = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

- Received data: 1010010 1110000 ...
  $\rightarrow w_1 = (1010010),\ w_2 = (1110000), \dots$
- Syndrome: $s_j = w_j \cdot \mathcal{H}_S^t$; $s_1 = (111), s_2 = (000)$
- The error vectors are $e_1 = (0001000)$ and $e_2 = (0000000)$.
- The corrected codewords are $v_1' = (1011010)$ and $v_2' = (1110000)$.

$\mathbb{Z}_2\mathbb{Z}_4$ coding. Example.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(7,4;5,3;5)$, $\Phi(\mathcal{C})$ is perfect, generated by

$$\mathcal{G}_S = \left(\begin{array}{ccccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right).$$

- Binary Information: $i_b = (10111110110)$
- Information (over $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$): $i = \Phi^{-1}(i_b) = (10111|213)$.

$\mathbb{Z}_2\mathbb{Z}_4$ coding. Example.

- $i = (10111|213)$

$$\chi(\mathcal{G}_S) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

- Codeword (over $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$): $\mathbf{v} = \chi^{-1}(\iota(i) \cdot \chi(\mathcal{G}_S)) =$
$= \chi^{-1}((10111213) \cdot \chi(\mathcal{G}_S)) = \chi^{-1}(20222222213) = (1011111|2213)$.
- Codeword (binary): $v_b = \Phi(\mathbf{v}) = 101111111110110$

$\mathbb{Z}_2\mathbb{Z}_4$ decoding: syndrome table

Let $C = \Phi(\mathcal{C})$ be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with error correcting capability t. Consider $\{e_i\}_{i=1}^r$ all error vectors with $w_t(e_i) \leq t$. Let $\mathcal{H}$ be the parity check matrix of $\mathcal{C}$.

error $\subseteq \mathbb{Z}_2^{\alpha+2\beta}$	syndrome $\subseteq \mathbb{Z}_4^{\bar{\gamma}+\bar{\delta}}$
$\mathbf{0}$	$\mathbf{0}$
e_1	$s_1 = \iota(\Phi^{-1}(e_1)) \cdot \chi(\mathcal{H})^t$
$\vdots$	$\vdots$
e_r	$s_r = \iota(\Phi^{-1}(e_r)) \cdot \chi(\mathcal{H})^t$

- For a received binary w, compute $s = \iota(\Phi^{-1}(w)) \cdot \chi(\mathcal{H})^t$.
- If $s = s_j$, then decode by $v' = w - e_j$.

$\mathbb{Z}_2\mathbb{Z}_4$ decoding. Example of a perfect code.

The parity-check matrix of $\mathcal{C}$ is

$$\mathcal{H}_S = \left(\begin{array}{ccccccc|cccc} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 3 & 3 \end{array}\right).$$

- Binary received vector: 10**0**11111110110.
- Received vector (over $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$): $\mathbf{w} = (1001111|2213)$.
- Syndrome: $s = \iota(\mathbf{w}) \cdot \chi(\mathcal{H}_S)^t = (222)$ is (+/-) a column in $\chi(\mathcal{H}_S)$.
- The error is $e = (0010000|0000)$ and the codeword is $\mathbf{v}' = (1011111|2213) \to$ binary codeword 101111111110110.

$\mathbb{Z}_2\mathbb{Z}_4$ decoding. Example of a perfect code.

The parity-check matrix of $\mathcal{C}$ is

$$\mathcal{H}_S = \left(\begin{array}{ccccccc|cccc} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 3 & 3 \end{array}\right).$$

- Binary received vector: 101111111110111
- Received vector (over $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$): $\mathbf{v} = (1011111|2212)$
- Syndrome: $s = \iota(\mathbf{v}) \cdot \chi(\mathcal{H}_S)^t = (021)$ is (+/-) a column in $\chi(\mathcal{H}_S)$.
- The error is $e = (0000000|0003)$ and the codeword is $\mathbf{v}' = (1011111|2213) \to$ binary codeword 101111111110110.

Permutation Decoding

$\mathbb{Z}_2\mathbb{Z}_4$-linear codes are also systematic codes and can be decoded by using permutation decoding.

[BBFV15] J. J. Bernal, J. Borges, C. Fernández-Córdoba, M. Villanueva.
Permutation Decoding of $\mathbb{Z}_2\mathbb{Z}_4$-linear Codes
Designs, Codes and Cryptography, vol. 76, pp. 269-277, 2015.

MAGMA. Computational Algebra System

http://magma.maths.usyd.edu.au/magma/
http://www.ccsg.uab.cat (Downloads/Z2Z4-Additive Codes version 4.0)

Some functions for $\mathbb{Z}_2\mathbb{Z}_4$-**additive codes**:

- `Z2Z4AdditiveCode(L : Alpha:=0, OverZ2:=false) List -> Rec`
- `Z2Z4Type(C) : Rec -> [ RngIntElt ]`
- `Z2Z4GeneratorMatrix(C) : Rec -> ModMatRngElt`
- `Z2Z4ParityCheckMatrix(C) : Rec -> ModMatRngElt`
- `Z2Z4MinRowsGeneratorMatrix(C) : Rec -> ModMatRngElt`
- `Z2Z4MinRowsParityCheckMatrix(C) : Rec -> ModMatRngElt`
- `Z2Z4StandardForm(C) : Rec -> Rec, Map, ModMatRngElt, GrpPermElt`
- `Z2Z4Dual(C) : Rec -> Rec`
- `Z2Z4DualType(C) : Rec -> [ RngIntElt ]`
- `IsZ2Z4SelfOrthogonal(C) : Rec -> BoolElt`
- `IsZ2Z4SelfDual(C) : Rec -> BoolElt`
- `Z2Z4GrayMap(C) : Rec -> Map`
- `Z2Z4GrayMapImage(C) : Rec -> [ ModTupRngElt ]`

3 $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes
- Classification of $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes
- Allowable α and β values
- Constructions of $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes

Bibliography

[BDF12] J. Borges, S. T. Dougherty, C. Fernández-Córdoba.
Characterization and Constructions of Self-Dual codes over $\mathbb{Z}_2 \times \mathbb{Z}_4$.
Advances in Mathematics of Communications, vol. 6, n. 3, pp. 287-303, 2012.

3 $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes

- Classification of $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes
- Allowable α and β values
- Constructions of $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes

Examples 16.

Consider the matrices

$$\mathcal{G}_1 = \left(\begin{array}{c|c} 1010 & 2000 \\ 0101 & 2000 \\ 0000 & 2200 \\ 0000 & 2020 \\ 0011 & 1111 \end{array} \right); \mathcal{G}_2 = \left(\begin{array}{c|c} 1010 & 00 \\ 0101 & 00 \\ 0000 & 20 \\ 0000 & 02 \end{array} \right).$$

The codes generated by these matrices are $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual. The code generated by $\mathcal{G}_1$ is non-separable and the code generated by $\mathcal{G}_2$ is separable.

The following theorem show some properties of separable $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes.

Theorem 12 (BDF12).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(2\kappa, \beta; \beta+\kappa-2\delta, \delta; \kappa)$. The following statements are equivalent:

(i) *$\mathcal{C}_X$ is a binary self-orthogonal code.*

(ii) *$\mathcal{C}_X$ is a binary self-dual code.*

(iii) $|\mathcal{C}_X| = 2^\kappa$.

(iv) *$\mathcal{C}_Y$ is a quaternary self-orthogonal code.*

(v) *$\mathcal{C}_Y$ is a quaternary self-dual code.*

(vi) $|\mathcal{C}_Y| = 2^\beta$.

(vii) *$\mathcal{C}$ is separable.*

Theorem 13 (BDF12).

If C is a binary self-dual code of length α and $\mathcal{D}$ is a quaternary self-dual code of length β, then $C \times \mathcal{D}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of length $\alpha + \beta$.

Antipodality

A binary code C is antipodal if for any codeword $z \in C$, $z + \mathbf{1} \in C$.
If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, we say that $\mathcal{C}$ is antipodal if $\Phi(\mathcal{C})$ is antipodal.
Clearly, a $\mathbb{Z}_2\mathbb{Z}_4$-additive code is antipodal iff $(\mathbf{1}^\alpha \mid \mathbf{2}^\beta) \in \mathcal{C}$.

Examples 17.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes generated by

$$\mathcal{G}_1 = \left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right); \mathcal{G}_2 = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right).$$

Both codes are $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual. The code $\mathcal{C}_1$ is non-antipodal and the code $\mathcal{C}_2$ is antipodal.

Type of a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual.

- If $\mathcal{C}$ has odd weights, then it is Type 0.
- If it has only even weights, then the $\mathcal{C}$ is Type I.
- If all the codewords have doubly-even weight, then $\mathcal{C}$ is Type II.

Examples 18.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes generated by

$$\mathcal{G}_1 = \left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{array} \right); \mathcal{G}_2 = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right).$$

The codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual. The code $\mathcal{C}_1$ is Type 0 and the code $\mathcal{C}_2$ is Type I.

Examples 19.

The code $\mathcal{C}_3$ generated by

$$\mathcal{G}_3 = \left(\begin{array}{c|c} 10001110 & 0000 \\ 01001101 & 0000 \\ 00101011 & 0000 \\ 00010111 & 0000 \\ 00000000 & 0202 \\ 00000000 & 2020 \\ 00000000 & 1111 \end{array}\right)$$

is $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual and of Type II.

Relationship among separability, antipodality and Type

The following table shows the relations among Type, separability and antipodality.

	Type 0	Type I	Type II
separability	non-separable	separable or non-separable	separable or non-separable
antipodality	non-antipodal	antipodal	antipodal

Now we will see some examples that show the existence of all possible cases described in the above table.

Type 0

Examples 20.

The code $\mathcal{C}_1$ generated by the matrix

$$\mathcal{G}_1 = \left(\begin{array}{c|c} 11 & 20 \\ 01 & 11 \end{array} \right)$$

is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of Type 0; the vector $(01|11)$ is an odd weight vector. Since it is Type 0, $\mathcal{C}_1$ is non-separable and non-antipodal.

Type I, separable

Examples 21.

Consider the code $\mathcal{C}_2$ generated by the matrix

$$\mathcal{G}_2 = \left(\begin{array}{c|c} 11 & 0 \\ 00 & 2 \end{array} \right).$$

Notice that for $\alpha = 2$ and $\beta = 1$, it is not possible to have odd weight codewords. Thus, the code must be of Type I and antipodal. Also, we have that the code restricted to the quaternary coordinates is $\{\mathbf{0}, \mathbf{2}\}$ which is self-dual and hence, $\mathcal{C}_2$ is separable.

Type I, non-separable

Examples 22.

Consider the following matrices:

$$\mathcal{G}_3 = \left(\begin{array}{c|c} 1111 & 0000 \\ 0101 & 2000 \\ 0101 & 0200 \\ 0101 & 0020 \\ 0011 & 1111 \end{array}\right); \quad \mathcal{G}_4 = \left(\begin{array}{c|c} 1111 & 000000 \\ 0101 & 220000 \\ 0000 & 202000 \\ 0101 & 000200 \\ 0101 & 111010 \\ 0011 & 101101 \end{array}\right). \tag{12}$$

The codes $\mathcal{C}_3$ and $\mathcal{C}_4$ generated by $\mathcal{G}_3$ and $\mathcal{G}_4$, respectively, are non-separable Type I.

Type II, separable

Examples 23.

As we have seen previously, the code defined in Example 19, generated by

$$\left(\begin{array}{c|c} 10001110 & 0000 \\ 01001101 & 0000 \\ 00101011 & 0000 \\ 00010111 & 0000 \\ 00000000 & 0202 \\ 00000000 & 2020 \\ 00000000 & 1111 \end{array}\right).$$

is $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual, separable and of Type II.

Type II, non-separable

Examples 24.

The code $\mathcal{C}_6$ generated by the following matrix

$$\left(\begin{array}{c|c} 10010110 & 0000 \\ 01001110 & 0000 \\ 00100111 & 0000 \\ 00000110 & 2000 \\ 00000110 & 0200 \\ 00000110 & 0020 \\ 00011011 & 1111 \end{array}\right).$$

is non-separable, since $(\mathcal{C}_6)_X$ is not self-orthogonal. On the other hand, it can be checked that all weights are doubly-even.

Allowable α and β values

Proposition 14 (BDF12).

There exist $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes of type $(\alpha, \beta; \gamma, \delta; \kappa)$ for all even α and all β.

Theorem 15.

If $\mathcal{C}$ is a Type II $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, then

$$\alpha \equiv 0 \pmod 8, \text{ and } \beta \equiv 0 \pmod 4.$$

Theorem 16 (BDF12).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, with $\alpha, \beta > 0$.

(i) *If $\mathcal{C}$ is Type 0, then $\alpha \geq 2$, $\beta \geq 2$.*

(ii) *If $\mathcal{C}$ is Type I and separable, then $\alpha \geq 2$, $\beta \geq 1$.*

(iii) *If $\mathcal{C}$ is Type I and non-separable, then $\alpha \geq 4$, $\beta \geq 4$.*

(iv) *If $\mathcal{C}$ is Type II, then $\alpha \geq 8$, $\beta \geq 4$.*

We define α_{min} and β_{min} to the minimum values of α and β for each Type of code and separability condition.

Example 25.

Type 0	Type I	Type II
–	$\left(\begin{array}{c\|c} 11 & 0 \\ 00 & 2 \end{array}\right);$	$\left(\begin{array}{cc} 10000111 & 0000 \\ 01001011 & 0000 \\ 00101101 & 0000 \\ 00011110 & 0000 \\ 00000000 & 2200 \\ 00000000 & 2020 \\ 00000000 & 1111 \end{array}\right).$
$\left(\begin{array}{c\|c} 11 & 20 \\ 01 & 11 \end{array}\right);$	$\left(\begin{array}{c\|c} 1010 & 2000 \\ 0101 & 2000 \\ 0101 & 0200 \\ 0101 & 0020 \\ 0011 & 1111 \end{array}\right);$	$\left(\begin{array}{c\|c} 10010110 & 0000 \\ 01001110 & 0000 \\ 00100111 & 0000 \\ 00000110 & 2000 \\ 00000110 & 0200 \\ 00000110 & 0020 \\ 00011011 & 1111 \end{array}\right).$

Theorem 17 (BDF12).

Let α_{min} and β_{min} be as defined above.

(i) *There exist a Type 0 or Type I code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ if and only if $\alpha = \alpha_{min} + 2a$, $a \geq 0$, $\beta \geq \beta_{min}$.*

(ii) *There exist a Type II code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ if and only if $\alpha = \alpha_{min} + 8a$, $\beta = \beta_{min} + 4b$, $a, b \geq 0$.*

The following table sumarizes the allowable values of α and β deppending on the Type of the code and the separability.

	Type 0	Type I	Type II
separable $\alpha, \beta; a, b \geq 0$	- -	$\alpha = 2 + 2a$ $\beta = 1 + b$	$\alpha = 8 + 8a$ $\beta = 4 + 4b$
non-separable $\alpha, \beta; a, b \geq 0$	$\alpha = 2 + 2a$ $\beta = 2 + b$	$\alpha = 4 + 2a$ $\beta = 4 + b$	$\alpha = 8 + 8a$ $\beta = 4 + 4b$

3. $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes
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Constructions of $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes

Three different constructions:

- Product of codes.
- Neighbor contruction.
- Extending the length.

Product of codes

Proposition 18 (BDF12).

If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and $\mathcal{D}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(\alpha', \beta'; \gamma', \delta'; \kappa')$ then $\mathcal{C} \times \mathcal{D}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(\alpha + \alpha', \beta + \beta'; \gamma + \gamma', \delta + \delta'; \kappa + \kappa')$.

Examples 26.

$$\mathcal{G}_C = \left(\begin{array}{c|c} 11 & 20 \\ 01 & 11 \end{array} \right); \qquad \mathcal{G}_D = \left(\begin{array}{c|c} 1010 & 2000 \\ 0101 & 2000 \\ 0101 & 0200 \\ 0101 & 0020 \\ 0011 & 1111 \end{array} \right);$$

$$\mathcal{G}_{CxD} = \left(\begin{array}{cc|cc} 11 & 0000 & 20 & 0000 \\ 01 & 0000 & 11 & 0000 \\ 00 & 1010 & 00 & 0000 \\ 00 & 0101 & 00 & 2000 \\ 00 & 0101 & 00 & 0200 \\ 00 & 0101 & 00 & 0020 \\ 00 & 0011 & 00 & 1111 \end{array} \right); \quad \mathcal{G}'_{CxD} = \left(\begin{array}{c|c} 100001 & 200000 \\ 010100 & 000000 \\ 001010 & 020000 \\ 001010 & 002000 \\ 001010 & 000200 \\ 000110 & 011110 \\ 000001 & 300001 \end{array} \right);$$

C is of type $(2,2;1,1;1)$, D is of type $(4,4;4,1;2)$ and $C \times D$ is of type $(5,5;5,2;3)$.

Neighbor construction

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code and let $\mathbf{v} \notin \mathcal{C}$ be a self-orthogonal vector. Let $\mathcal{C}_\mathbf{v}$ be the subcode of $\mathcal{C}$ of vectors orthogonal to $\mathbf{v}$

$$\mathcal{C}_\mathbf{v} = \{\mathbf{u} \in \mathcal{C} \mid \mathbf{u} \cdot \mathbf{v} = 0\}.$$

Theorem 19 (BDF12).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code and let $\mathbf{v}$ be a self-orthogonal vector that is not an element of $\mathcal{C}$. Then

$$N(\mathcal{C}, \mathbf{v}) = \langle \mathcal{C}_\mathbf{v}, \mathbf{v} \rangle$$

is a self-dual code.

Examples 27.

Let $\mathcal{C}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by the matrix

$$\mathcal{G} = \left(\begin{array}{c|c} 11 & 20 \\ 01 & 11 \end{array} \right),$$

and let $\mathbf{v} = (00|20)$.
$\mathcal{C} = \{(00|00), (11|20), (01|11), (00|22), (01|33), (10|31), (11|02), (10|13)\}$.
Then, $\mathcal{C}_{\mathbf{v}} = \{(00|00), (11, |20), (00|22), (11|02)\}$, is generated by

$$\mathcal{G}_{\mathbf{v}} = \left(\begin{array}{c|c} 11 & 20 \\ 00 & 22 \end{array} \right),$$

Then, the code $N(\mathcal{C}, \mathbf{v}) = \langle \mathcal{C}_{\mathbf{v}}, \mathbf{v} \rangle$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code generated by

$$\mathcal{G}_{N(\mathcal{C},\mathbf{v})} = \left(\begin{array}{c|c} 11 & 00 \\ 00 & 20 \\ 00 & 02 \end{array} \right)$$

Extending the length

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code, $\mathbf{v} \notin \mathcal{C}$. $\mathcal{C}_\mathbf{v}$ is a subgroup of $\mathcal{C}$ and $\mathcal{C}_\mathbf{v}^\perp = \langle \mathcal{C}, \mathbf{v} \rangle$. Moreover,

$$\frac{|\mathcal{C}|}{|\mathcal{C}_\mathbf{v}|} = \frac{|\mathcal{C}_\mathbf{v}^\perp|}{|\mathcal{C}|} \in \{2, 4\}.$$

Let $\mathbf{w}$ be the vector such that $\mathcal{C} = \langle \mathcal{C}_\mathbf{v}, \mathbf{w} \rangle$. Then

$$\mathcal{C}_\mathbf{v}^\perp = \langle \mathcal{C}_\mathbf{v}, \mathbf{w}, \mathbf{v} \rangle.$$

Examples 28.

Let $\mathcal{C}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by the matrix

$$\mathcal{G} = \left(\begin{array}{c|c} 11 & 20 \\ 01 & 11 \end{array} \right),$$

and let $\mathbf{v} = (00|20)$ as in Example 27. Then $\mathcal{C}_\mathbf{v}$ is generated by

$$\mathcal{G}_\mathbf{v} = \left(\begin{array}{c|c} 11 & 20 \\ 00 & 22 \end{array} \right),$$

and $\mathcal{C} = \langle \mathcal{C}_\mathbf{v}, \mathbf{w} \rangle$, where $\mathbf{w} = (01|11)$. The code $\mathcal{C}_\mathbf{v}^\perp$ is generated by

$$\mathcal{H}_\mathbf{v} = \left(\begin{array}{c|c} 11 & 20 \\ 00 & 22 \\ 00 & 02 \\ 01 & 11 \end{array} \right).$$

Construction of $\bar{\mathcal{D}}$ by extending the length of $\mathcal{D} = \mathcal{C}_v^\perp$

For $\mathbf{u} = (u_X, u_Y) \in \mathcal{C}_\mathbf{v}^\perp$ we define the extension of $\mathbf{u}$ as

$$\bar{\mathbf{u}} = (u'_X, u_X, u_Y, u'_Y).$$

If $\mathbf{u} \in \mathcal{C}_\mathbf{v}$, then $\bar{\mathbf{u}} = (\mathbf{0}, u_X, u_Y, \mathbf{0})$.
Then

$$\bar{\mathcal{D}} = \langle \{\bar{\mathbf{u}} \mid \mathbf{u} \in \mathcal{C}_\mathbf{v}^\perp\} \rangle.$$

We choose u'_X and u'_Y so that $\bar{\mathcal{D}}$ is a self-orthogonal code. If $\bar{\mathcal{D}}$ is not self-dual we may need to add additional vectors to the code.

Theorem 20 (BDF12).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and $\mathbf{v} \notin \mathcal{C}$. Let $\mathbf{w}$, $\mathcal{C}_\mathbf{v}$ be as before and $\mathcal{D} = \mathcal{C}_\mathbf{v}^\perp = \langle \mathcal{C}_\mathbf{v}, \mathbf{w}, \mathbf{v} \rangle$. There exists a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code $\langle \bar{\mathcal{D}}, V \rangle$ of type $(\alpha + \alpha', \beta + \beta'; \gamma', \delta'; \kappa')$, for some set of vectors V with the following conditions:

(i) $\alpha' \neq 0$ *and* $\beta' = 0$ *only if* $\mathbf{v} \cdot \mathbf{w} = 2$ *and* $\mathbf{v} \cdot \mathbf{v} \in \{0, 2\}$,

(ii) $\alpha' = 0$ *and* $\beta' \neq 0$ *only if* $\mathbf{v} \cdot \mathbf{w} = 2$ *or* $\mathbf{v} \cdot \mathbf{w} \in \{1, 3\}$ *and* $\mathbf{v} \cdot \mathbf{v} \in \{1, 3\}$,

(iii) $\alpha' \neq 0$ *and* $\beta' \neq 0$.

Table: Case $\alpha' \neq 0, \beta' = 0$.

$\mathbf{v}\cdot\mathbf{v}$	v'_X	w'_X	V
0	$(0,0,1,1)$	$(0,1,0,1)$	$\{(1,1,1,1,\mathbf{0})\}$
2	$(0,1)$	$(1,1)$	$\emptyset$

Table: Case $\alpha' = 0, \beta' \neq 0$, $\mathbf{v}\cdot\mathbf{w} = 2$.

$\mathbf{v}\cdot\mathbf{v}$	v'_Y	w'_Y	V
0	$(1,1,1,1)$	$(2,0,0,0)$	$\{(\mathbf{0},0,2,2,0),(\mathbf{0},0,0,2,2)\}$
1	$(1,1,1)$	$(2,0,0)$	$\{(\mathbf{0},0,2,2)\}$
2	$(1,1)$	$(2,0)$	$\emptyset$
3	(1)	(2)	$\emptyset$

Table: Case $\alpha' = 0, \beta' \neq 0$, $\mathbf{v}\cdot\mathbf{w} = 1$.

$\mathbf{v}\cdot\mathbf{v}$	v'_Y	w'_Y	V
1	$(1,1,1,0)$	$(1,1,1,1)$	$\{(\mathbf{0},0,2,2,0),(\mathbf{0},2,2,0,0)\}$
3	$(3,0,0,0)$	$(1,1,1,1)$	$\{(\mathbf{0},0,2,2,0),(\mathbf{0},0,0,2,2)\}$

Table: Case $\alpha' \neq 0, \beta' \neq 0$, $\mathbf{v}\cdot\mathbf{w} = 1$.

$\mathbf{v}\cdot\mathbf{v}$	v'_X	v'_Y	w'_X	w'_Y	V
0	$(1,0)$	$(1,0,1)$	$(1,0)$	$(1,1,0)$	$\{(1,1,\mathbf{0},2,0,0),(1,1,\mathbf{0},0,2,0)\}$
1	$(1,0)$	$(1,0)$	$(1,0)$	$(1,1)$	$\{(1,1,\mathbf{0},2,0)\}$
2	$(1,1)$	$(0,1,1)$	$(1,0)$	$(1,1,0)$	$\{(1,1,\mathbf{0},2,0,0),(1,1,\mathbf{0},0,2,2)\}$
3	$(1,1)$	$(1,0)$	$(1,0)$	$(1,1)$	$\{(1,1,\mathbf{0},0,2)\}$

Table: Case $\alpha' \neq 0, \beta' \neq 0$, $\mathbf{v} \cdot \mathbf{w} = 2$.

$\mathbf{v} \cdot \mathbf{v}$	v'_X	v'_Y	w'_X	w'_Y	V
0	$(1,0)$	$(1,1)$	$(1,1)$	$(2,2)$	$\{(1,1,\mathbf{0},2,0)\}$
1	$(1,0)$	$(1,0)$	$(1,1)$	$(0,2)$	$\{(1,1,\mathbf{0},2,0)\}$
2	$(1,1)$	$(1,3)$	$(1,0)$	$(1,1)$	$\emptyset$
3	$(0,0)$	$(0,1)$	$(1,1)$	$(0,2)$	$\{(1,1,\mathbf{0},2,0)\}$

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. To construct a $\mathbb{Z}_2\mathbb{Z}_4$-additive code D of type $(\alpha + \alpha', \beta + \beta'; \gamma', \delta'; \kappa')$:

1) Select $\mathbf{v} \notin \mathcal{C}$ such that $\mathbf{v} \cdot \mathbf{v}$ is the appropppriate value given in the previous tables.
2) Construct $\mathcal{C}_{\mathbf{v}}$ and determine $\mathbf{w}$ such that $\mathcal{C} = \langle \mathcal{C}_{\mathbf{v}}, \mathbf{w} \rangle$.
3) From previous tables, determine the values of $v'_X, v'_Y, w'_X, w'_Y, V$.
4) Define $\mathcal{D} = \mathcal{C}_{\mathbf{v}}^{\perp} = \langle \mathcal{C}_{\mathbf{v}}, \mathbf{w}, \mathbf{v} \rangle$. If $\mathcal{G}_{\mathbf{v}}$ is the generator matrix of $\mathcal{C}_{\mathbf{v}}$, then, the generator matrix of $\bar{\mathcal{D}}$ is:

$$\mathcal{G}_{\bar{\mathcal{D}}} = \begin{pmatrix} \mathbf{0} & \mathcal{G}_{\mathbf{v}} & \mathbf{0} \\ v'_X & \mathbf{v} & v'_Y \\ w'_X & \mathbf{w} & w'_Y \\ & V & \end{pmatrix}.$$

Example 29.

Let $\mathcal{C}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by the matrix

$$\mathcal{G} = \left(\begin{array}{c|c} 11 & 20 \\ 01 & 11 \end{array} \right).$$

We want to extend the binary and also the quaternary coordinates. From Theorem 20, there is no restriction to $\mathbf{v}$ and $\mathbf{w}$.
Let $\mathbf{v} = (00|20)$, $\mathbf{w} = (01|11)$ and, by Example 28,

$$\mathcal{G}_{\mathbf{v}} = \left(\begin{array}{c|c} 11 & 20 \\ 00 & 22 \end{array} \right).$$

Note that $\mathbf{v} \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 2$

Example 30.

Table: Case $\alpha' \neq 0, \beta' \neq 0$, $\mathbf{v} \cdot \mathbf{w} = 2$.

$\mathbf{v} \cdot \mathbf{v}$	v'_X	v'_Y	w'_X	w'_Y	V
0	$(1,0)$	$(1,1)$	$(1,1)$	$(2,2)$	$\{(1,1,\mathbf{0},2,0)\}$

The generator matrix of $\bar{\mathcal{D}}$ is:

$$\mathcal{G}_{\bar{\mathcal{D}}} = \begin{pmatrix} \mathbf{0} & \mathcal{G}_{\mathbf{v}} & \mathbf{0} \\ v'_X & \mathbf{v} & v'_Y \\ w'_X & \mathbf{w} & w'_Y \\ & V & \end{pmatrix} = \left(\begin{array}{cc|cc} 00 & 11 & 20 & 00 \\ 00 & 00 & 22 & 00 \\ 10 & 00 & 20 & 11 \\ 11 & 01 & 11 & 22 \\ 11 & 00 & 00 & 20 \end{array}\right).$$

$\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. We say that $\mathcal{C}$ is $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual if $W_{\mathcal{C}^\perp}(x, y) = W_{\mathcal{C}}(x, y)$.

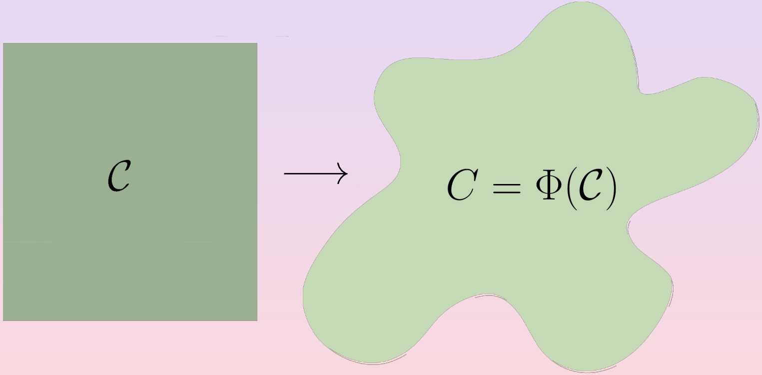
[DF14] S. T. Dougherty, C. Fernández-Córdoba.
$\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual codes
Designs, Codes and Cryptography, vol. 72, pp. 435-453, 2014.

Bibliography

[FPV10] C. Fernández-Córdoba, J.Pujol, M. Villanueva.
$\mathbb{Z}_2\mathbb{Z}_4$-linear codes: rank and kernel.
Designs, Codes and Cryptography, vol. 56, pp. 43-59, 2010.

[BDFT19] J. Borges, S. T. Dougherty, C. Fernández-Córdoba, R. Ten-Valls.
$\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes: kernel and rank.
IEEE Transactions on Information Theory, vol. 65, pp. 2119-2127, 2019.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code and $C = \Phi(\mathcal{C})$.



Example 31.

Consider the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}_{15}$ of type $(3,5;3,3;3)$ generated by the following matrix:

$$\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \left(\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right). \qquad (13)$$

$C_{15} = \Phi(\mathcal{C}_{15})$ is not linear: $\Phi(\mathbf{v}_2) + \Phi(\mathbf{v}_3) \notin C_{15}$;

$$\Phi^{-1}(\Phi(\mathbf{v}_2) + \Phi(\mathbf{v}_3)) = \Phi^{-1}((000\,0001000100) + (000\,0001000001)) = \Phi^{-1}(000\,0000000101) = (000 \mid 00011) \notin \mathcal{C}_{15}.$$

Definitions of rank and kernel

Let C be a binary code, $\mathbf{0} \in C$.

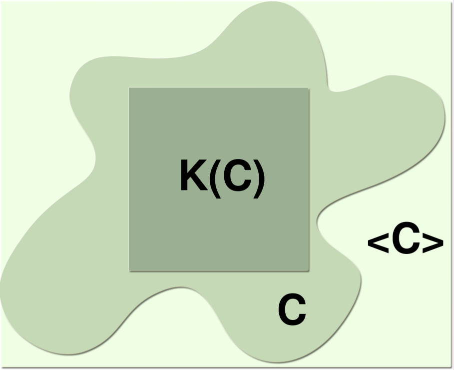
- Rank of C: $rank(C) = dim\langle C \rangle$.
- Kernel of C: $K(C) = \{x \in C \mid C = C + x\}$,
  $ker(C) = dim(K(C))$.

$$K(C) = \bigcap_{i \in \{0,\cdots,s\}} D_i,$$

where $D_0 \ldots, \mathcal{D}_s$ are all the maximal linear subspaces of C [PL95].

[PL95] K. T. Phelps, M. Levan.
Kernels of nonlinear Hamming codes
Designs, Codes and Cryptography, vol. 6, pp. 247-257, 1995.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code and $C = \Phi(\mathcal{C})$.



$$K(C) \subseteq C \subseteq \langle C \rangle.$$

Why do we study rank and kernel?

Let C_i be a binary code, with rank r_i and dimension of the kernel k_i for $i \in \{1, 2\}$.

- If C_i is linear, then $K(C_i) = C_i = \langle C_i \rangle$.

- If $r_1 \neq r_2$, then C_1 is not equivalent to C_2.
- If $k_1 \neq k_2$, then C_1 is not equivalent to C_2.

- $C_i = \bigcup_{j \in \{0, \cdots, t\}} K(C_i) + v_j$, where $v_0 = \mathbf{0}, v_1, \cdots, v_t$ are coset representatives.

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- $C_i = \bigcup_{j \in \{0, \cdots, t\}} K(C_i) + v_j$, where $v_0 = \mathbf{0}, v_1, \cdots, v_t$ are coset representatives.

Linearity

Let $\mathbf{u} = (u_1, \ldots, u_{\alpha+\beta}), \mathbf{v} = (v_1, \ldots, v_{\alpha+\beta}) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$.

$$\mathbf{u} * \mathbf{v} = (u_1 v_1, \ldots, u_{\alpha+\beta} v_{\alpha+\beta}).$$

Proposition 21.

Let $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. *Then,* $\Phi(\mathbf{u}+\mathbf{v}) = \Phi(\mathbf{u}) + \Phi(\mathbf{v}) + \Phi(2\mathbf{u} * \mathbf{v})$.

Linearity

Corollary 22 (FPV10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. Then, $C = \Phi(\mathcal{C})$ is linear if and only if $2\mathbf{u} * \mathbf{v} \in \mathcal{C}$ $\forall \mathbf{u}, \mathbf{v} \in \mathcal{C}$.

Note that if $\mathbf{u} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is of order two, then $2\mathbf{u} \star \mathbf{v} = \mathbf{0} \in \mathcal{C}$, for all $\mathbf{v} \in \mathcal{C}$.

Linearity

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Proposition 23 (FPV10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix $\mathcal{G}$. Let $\{\mathbf{u}_i\}_{i=1}^{\gamma}$ and $\{\mathbf{v}_j\}_{j=1}^{\delta}$ be the row vectors of order two and four in $\mathcal{G}$, respectively. Then, $C = \Phi(\mathcal{C})$ is linear if and only if $2\mathbf{v}_j * \mathbf{v}_k \in \mathcal{C}$ for all j, k satisfying $1 \leq j < k \leq \delta$.

Corollary 24 (FPV10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. If $\delta \leq 1$, then $\Phi(\mathcal{C})$ is linear.

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Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. If $\delta \leq 1$, then $\Phi(\mathcal{C})$ is linear.

Example 32.

Consider the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}_{15}$ of type $(3,5;3,3;3)$ generated by the following matrix:

$$(\mathcal{G}_{15})_S = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \left(\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

$\Phi(C_{15})$ is not linear; $2\mathbf{v}_2 * \mathbf{v}_3 = (000 \mid 02000) \notin \mathcal{C}_{15}$.

Example 33.

Consider the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ of type $(3,3;3,2;3)$ generated by the following matrix:

$$\mathcal{G}_S = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right). \quad (14)$$

$C = \Phi(\mathcal{C})$ is linear: for all $\mathbf{v}_i, \mathbf{v}_j \in \mathcal{C}$, $1 \leq j < k \leq \delta$, $2\mathbf{v}_i * \mathbf{v}_j \in \mathcal{C}$; that is, $2\mathbf{v}_1 * \mathbf{v}_2 = \mathbf{0} \in \mathcal{C}$.

Lemma 25.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. If $\Phi(\mathcal{C})$ is linear, then $\phi(\mathcal{C}_Y)$ is linear.

The converse is not true in general.

Proposition 26 (BDFT19).

Let $\mathcal{C}$ be a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive code. Then, $\Phi(\mathcal{C})$ is linear if and only if $\phi(\mathcal{C}_Y)$ is linear.

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Example 34.

Let $\mathcal{C}_{15}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive code given in Example 32. We have seen that $\Phi(\mathcal{C}_{15})$ is not linear.

$$(\mathcal{G}_{15})_S = \begin{pmatrix} (u_1 \mid u_1') \\ (u_2 \mid u_2') \\ (u_3 \mid u_3') \\ (v_1 \mid v_1') \\ (v_2 \mid v_2') \\ (v_3 \mid v_3') \end{pmatrix} = \left(\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right). \qquad (15)$$

- $2v_1' * v_2' = 2v_1' * v_3' = \mathbf{0} \in (\mathcal{C}_{15})_Y$,
- $2v_2' * v_3' = (0, 2, 0, 0, 0) \in (\mathcal{C}_{15})_Y$.

Then, $\phi((\mathcal{C}_{15})_Y)$ is linear.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix in standard form,

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_\kappa & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S_b & S_q & R & I_\delta \end{array} \right),$$

and let $\mathcal{C}'$ be the subcode generated by

$$\mathcal{G}' = \left(\begin{array}{cc|ccc} \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S_b & S_q & R & I_\delta \end{array} \right). \qquad (16)$$

$$\mathcal{G}' = \left(\begin{array}{cc|ccc} \not{I_\kappa} & \not{T_b} & \not{2T_2} & \not{\mathbf{0}} & \not{\mathbf{0}} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S_b & S_q & R & I_\delta \end{array}\right).$$

Proposition 27 (BDFT19).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix in standard form, and let $\mathcal{C}'$ be the subcode generated by $\mathcal{G}'$. Then, $\Phi(\mathcal{C})$ is linear if and only if $\phi(\mathcal{C}'_Y)$ is linear.

Example 35.

Let $\mathcal{C}_{15}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive code given in Example 32 genrated by $(\mathcal{C}_{15})_S$. We have seen that $\Phi(\mathcal{C}_{15})$ is not linear. Let

$$\mathcal{G}'_{15} = \begin{pmatrix} \not(u_1 \mid u'_1) \\ \not(u_2 \mid u'_2) \\ \not(u_3 \mid u'_3) \\ (v_1 \mid v'_1) \\ (v_2 \mid v'_2) \\ (v_3 \mid v'_3) \end{pmatrix} = \left(\begin{array}{ccc|ccccc} \not1 & \not0 & \not0 & \not0 & \not0 & \not0 & \not0 & \not0 \\ \not0 & \not1 & \not0 & \not2 & \not0 & \not0 & \not0 & \not0 \\ \not0 & \not0 & \not1 & \not0 & \not2 & \not0 & \not0 & \not0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

- $2v'_1 * v'_2 = 2v'_1 * v'_3 = \mathbf{0} \in (\mathcal{C}'_{15})_Y$,
- $2v'_2 * v'_3 = (0,2,0,0,0) \notin (\mathcal{C}'_{15})_Y$.

Then, $\phi((\mathcal{C}_{15})'_Y)$ is linear.

Rank of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. We define the code the $\mathbb{Z}_2\mathbb{Z}_4$-additive code

$$\mathcal{R}(\mathcal{C}) = \Phi^{-1}(\langle \Phi(\mathcal{C}) \rangle).$$

Proposition 28 (FPV10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Let $\mathcal{G}$ be a generator matrix of $\mathcal{C}$, and let $\{\mathbf{u}_i\}_{i=1}^{\gamma}$ be the rows of order two and $\{\mathbf{v}_j\}_{j=1}^{\delta}$ the rows of order four in $\mathcal{G}$. Then,

$$\mathcal{R}(\mathcal{C}) = \langle \mathcal{C}, \{2\mathbf{v}_j * \mathbf{v}_k\}_{1 \leq j < k \leq \delta} \rangle =$$

$$\langle \{\mathbf{u}_i\}_{i=1}^{\gamma}, \{\mathbf{v}_j\}_{j=1}^{\delta}, \{2\mathbf{v}_j * \mathbf{v}_k\}_{1 \leq j < k \leq \delta} \rangle.$$

Corollary 29 (FPV10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $\mathcal{R}(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma + \bar{r}, \delta; \kappa)$, with $\bar{r} \geq 0$, and $\mathrm{rank}(\Phi(\mathcal{C})) = \log_2(|\mathcal{R}(\mathcal{C})|) = \gamma + 2\delta + \bar{r}$.

Corollary 30 (FPV10).

If C is a $\mathbb{Z}_2\mathbb{Z}_4$-linear code, then $\langle C \rangle$ is both linear and $\mathbb{Z}_2\mathbb{Z}_4$-linear.

Example 36.

Consider the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}_{15}$ of type $(3,5;3,3;3)$ generated by the following matrix:

$$(\mathcal{G}_{15})_S = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \left(\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

Note that $2\mathbf{v}_1 * \mathbf{v}_2 = 2\mathbf{v}_1 * \mathbf{v}_3 = \mathbf{0} \in \mathcal{C}_{15}$.
$\mathcal{R}(\mathcal{C}_{15}) = \langle \mathcal{C}_{15}, 2\mathbf{v}_2 * \mathbf{v}_3 \rangle = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, 2\mathbf{v}_2 * \mathbf{v}_3 \rangle$.

Example 37.

$\mathcal{C}_{15}$ of type $(3,5;3,3;3)$. We have that $\mathcal{R}(C_{15})$ is generated by

$$(\mathcal{G}_{15})_S = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 - (2\mathbf{v}_2 \star \mathbf{v}_3) \\ (2\mathbf{v}_2 \star \mathbf{v}_3) \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \left(\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

$$rank(\mathcal{C}_{15}) = \gamma + 2\delta + 1 = 3 + 2 \cdot 3 + 1 = 10.$$

If $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$, then it is easy to see that $2(u \mid u') \star (v \mid v') \in \mathcal{C}$ if and only if $2u' \star v' \in \mathcal{C}_Y$.

Proposition 31 (BDFT19).

If $\mathcal{C}$ is a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, then $\mathcal{R}(\mathcal{C}) = \mathcal{C}_X \times \mathcal{R}(\mathcal{C}_Y)$ and $\mathrm{rank}(\Phi(\mathcal{C})) = \kappa + \mathrm{rank}(\phi(\mathcal{C}_Y))$.

If $\mathcal{C}$ is not separable, then it is not true in general.

If $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$, then it is easy to see that $2(u \mid u') \star (v \mid v') \in \mathcal{C}$ if and only if $2u' \star v' \in \mathcal{C}_Y$.

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If $\mathcal{C}$ is not separable, then it is not true in general.

Example 38.

Consider the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}_{15}$ of type $(3,5;3,3;3)$ generated by the following matrix:

$$(\mathcal{G}_{15})_S = \left(\begin{array}{c} (u_1 \mid u_1') \\ (u_2 \mid u_2') \\ (u_3 \mid u_3') \\ (v_1 \mid v_1') \\ (v_2 \mid v_2') \\ (v_3 \mid v_3') \end{array} \right) = \left(\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right).$$

$\mathcal{R}(\mathcal{C}_{15}) = \langle \mathcal{C}_{15}, 2\mathbf{v}_2 * \mathbf{v}_3 \rangle = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, 2\mathbf{v}_2 * \mathbf{v}_3 \rangle$.
We have seen that $(\mathcal{C}_{15})_Y$ is linear, so

$$\mathcal{R}((\mathcal{C}_{15})_Y) = \langle u_2', u_3', v_1', v_2', v_3' \rangle.$$

$$\mathcal{G}' = \left(\begin{array}{cc|ccc} \cancel{I_\kappa} & \cancel{T_b} & \cancel{2T_2} & \cancel{\mathbf{0}} & \cancel{\mathbf{0}} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S_b & S_q & R & I_\delta \end{array} \right).$$

Theorem 32 (BDFT19).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix in standard form, and let $\mathcal{C}'$ be the subcode generated by $\mathcal{G}'$. Then,

$$\mathrm{rank}(\Phi(\mathcal{C})) = \kappa + \mathrm{rank}(\phi(\mathcal{C}'_Y)).$$

Example 39.

Let $\mathcal{C}_{15}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive code given in Example 32 genrated by $(\mathcal{G}_{15})_S$. Let

$$(\mathcal{G}'_{15})_S = \begin{pmatrix} \cancel{(u_1 \mid u'_1)} \\ \cancel{(u_2 \mid u'_2)} \\ \cancel{(u_3 \mid u'_3)} \\ (v_1 \mid v'_1) \\ (v_2 \mid v'_2) \\ (v_3 \mid v'_3) \end{pmatrix} = \left(\begin{array}{ccc|ccccc} \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{1} & \cancel{0} & \cancel{2} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{2} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

$\mathcal{R}(\mathcal{C}_{15}) = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, 2\mathbf{v}_2 * \mathbf{v}_3 \rangle$; $rank(\Phi(\mathcal{C}_{15})) = 10$.
$\mathcal{R}((\mathcal{C}'_{15})_Y) = \langle v'_1, v'_2, v'_3, 2v'_2 \star v'_3 \rangle$; $\mathcal{R}((\mathcal{C}'_{15})_Y) = 7$

$$\mathcal{R}(\mathcal{C}_{15}) = \kappa + \mathcal{R}((\mathcal{C}'_{15})_Y).$$

Bounds for the rank of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

Proposition 33 (FPV10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $\mathcal{R}(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma + \bar{r}, \delta; \kappa)$, with $\bar{r} \geq 0$, and $rank(\Phi(\mathcal{C})) = \log_2(|\mathcal{R}(\mathcal{C})|) = \gamma + 2\delta + \bar{r}$, where

$$\bar{r} \in \left\{0, \ldots, \min\left\{\beta - (\gamma - \kappa) - \delta, \binom{\delta}{2}\right\}\right\}.$$

Theorem 34 (FPV10).

Let $\alpha, \beta, \gamma, \delta, \kappa$ be allowable parameters. Then, there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and rank $r = \gamma + 2\delta + \bar{r}$, for any

$$\bar{r} \in \left\{0, \ldots, \min\left\{\beta - (\gamma - \kappa) - \delta,\ \binom{\delta}{2}\right\}\right\}.$$

Example 40.

- Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of type $(\alpha, 5; 2, 3; 1)$. Then, $r = 8 + \bar{r}$, $\bar{r} \in \{0, \ldots, \min(1, 3)\} = \{0, 1\}$; $\quad r \in \{8, 9\}$.
- Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of type $(\alpha, 8; 2, 3; 1)$. Then, $r = 8 + \bar{r}$, $\bar{r} \in \{0 \ldots, \min(4, 3)\} = \{0, 1, 2, 3\}$; $r \in \{8, 9, 10, 11\}$.

Example

For $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}$ of type $(\alpha, 8; 2, 3; 1)$, $r \in \{8, 9, 10, 11\}$.

We obtain $\mathbb{Z}_2\mathbb{Z}_4$-linear codes $C = \Phi(\mathcal{C})$ for all possible ranks, taking the following generator matrix:

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} 1 & T_b & \mathbf{0} & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} \\ \hline \mathbf{0} & S_b & S_q & \mathbf{0} & I_3 \end{array}\right)$$

$$r = 8,\ \text{when } S_q = (\mathbf{0}) \qquad r = 9,\ \text{when } S_q = A$$

$$r = 10,\ \text{when } S_q = B \qquad r = 11,\ \text{when } S_q = C$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

4 Linearity, Rank and Kernel

- Basic definitions
- Linearity
- Rank of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes
- Kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes
- Pairs of rank and dimension of the kernel

Kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. We define the kernel of $\mathcal{C}$, denoted by $\mathcal{K}(\mathcal{C})$, as the $\mathbb{Z}_2\mathbb{Z}_4$-additive code

$$\mathcal{K}(\mathcal{C}) = \Phi^{-1}(K(\Phi(\mathcal{C}))).$$

Proposition 35 (FPV10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix $\mathcal{G}$. Let $\{\mathbf{u}_i\}_{i=1}^{\gamma}$ and $\{\mathbf{v}_j\}_{j=1}^{\delta}$ be the row vectors of order two and four in $\mathcal{G}$, respectively. Then,

$$\mathcal{K}(\mathcal{C}) = \{\mathbf{u} \in \mathcal{C} \mid 2\mathbf{u} * \mathbf{v}_j \in \mathcal{C}, \forall j \in \{1, \dots, \delta\}\}.$$

Corollary 36 (FPV10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. We have that

$$\mathcal{C}_b \subseteq \mathcal{K}(\mathcal{C}).$$

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix $\mathcal{G}$. Let $\{\mathbf{u}_i\}_{i=1}^{\gamma}$ and $\{\mathbf{v}_j\}_{j=1}^{\delta}$ be the row vectors of order two and four in $\mathcal{G}$, respectively. Then,

$$\langle \{\mathbf{u}_i\}_{i=1}^{\gamma}, \{2\mathbf{v}_j\}_{j=1}^{\delta} \rangle \subseteq \mathcal{K}(\mathcal{C}).$$

Corollary 36 (FPV10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. We have that

$$\mathcal{C}_b \subseteq \mathcal{K}(\mathcal{C}).$$

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix $\mathcal{G}$. Let $\{\mathbf{u}_i\}_{i=1}^{\gamma}$ and $\{\mathbf{v}_j\}_{j=1}^{\delta}$ be the row vectors of order two and four in $\mathcal{G}$, respectively. Then,

$$\langle \{\mathbf{u}_i\}_{i=1}^{\gamma}, \{2\mathbf{v}_j\}_{j=1}^{\delta} \rangle \subseteq \mathcal{K}(\mathcal{C}).$$

Example 41.

Consider the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}_{15}$ of type $(3,5;3,3;3)$ generated by the following matrix:

$$(\mathcal{G}_{15})_S = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \left(\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right). \qquad (17)$$

- $\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, 2\mathbf{v}_1, 2\mathbf{v}_2, 2\mathbf{v}_3 \rangle \subseteq \mathcal{K}(\mathcal{C}_{15})$.
- $2\mathbf{v}_1 * \mathbf{v}_2 = 2\mathbf{v}_1 * \mathbf{v}_3 = \mathbf{0} \in \mathcal{C}_{15}$; $\mathbf{v}_1 \in \mathcal{K}(\mathcal{C}_{15})$.
- $2\mathbf{v}_2 * \mathbf{v}_3 \notin \mathcal{C}_{15}$; $\mathbf{v}_2, \mathbf{v}_3, \notin \mathcal{K}(\mathcal{C}_{15})$.

$$\mathcal{K}(\mathcal{C}_{15}) = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, 2\mathbf{v}_2, 2\mathbf{v}_3 \rangle;\ \ker(\mathcal{C}_{15}) = 7.$$

Proposition 37 (BDFT19).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $\mathcal{K}(\mathcal{C}) \subseteq \mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y)$ and $\ker(\Phi(\mathcal{C})) \leq \kappa + \ker(\phi(\mathcal{C}_Y))$.

Proposition 38 (BDFT19).

If $\mathcal{C}$ is a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, then $\mathcal{K}(\mathcal{C}) = \mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y)$ and $\ker(\Phi(\mathcal{C})) = \kappa + \ker(\phi(\mathcal{C}_Y))$.

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Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha,\beta;\gamma,\delta;\kappa)$. Then, $\mathcal{K}(\mathcal{C}) \subseteq \mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y)$ and $\ker(\Phi(\mathcal{C})) \leq \kappa + \ker(\phi(\mathcal{C}_Y))$.

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If $\mathcal{C}$ is a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha,\beta;\gamma,\delta;\kappa)$, then $\mathcal{K}(\mathcal{C}) = \mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y)$ and $\ker(\Phi(\mathcal{C})) = \kappa + \ker(\phi(\mathcal{C}_Y))$.

Example 42.

Let $\mathcal{C}_{15}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(3,5;3,3;3)$ given in Example 32 genrated by $(\mathcal{G}_{15})_S$. Let

$$(\mathcal{G}'_{15})_S = \begin{pmatrix} \cancel{(u_1 \mid u'_1)} \\ \cancel{(u_2 \mid u'_2)} \\ \cancel{(u_3 \mid u'_3)} \\ (v_1 \mid v'_1) \\ (v_2 \mid v'_2) \\ (v_3 \mid v'_3) \end{pmatrix} = \left(\begin{array}{ccc|ccccc} \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{1} & \cancel{0} & \cancel{2} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{2} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

$\mathcal{K}(\mathcal{C}_{15}) = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, 2\mathbf{v}_2, 2\mathbf{v}_3 \rangle$; $\ker(\mathcal{C}_{15}) = 7$.
$\mathcal{K}(\mathcal{C}'_{15}) = \langle v'_1, 2v'_2, 2v'_3 \rangle$; $\ker(\mathcal{C}_{15}) = 4$.

$$\mathcal{K}(\mathcal{C}_{15}) = \kappa + \mathcal{K}(\mathcal{C}'_{15}).$$

Bounds on the kernel dimension of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

Proposition 39 (FPV10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $\mathcal{K}(C)$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive subcode of $\mathcal{C}$ of type $(\alpha, \beta; \gamma + \bar{k}, \delta - \bar{k}; \kappa)$ and $\ker(\Phi(C)) = \gamma + 2\delta - \bar{k}$, where $\bar{k} \in \{0\} \cup \{2, \ldots, \delta\}$.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$code with generator matrix:

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_\kappa & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S_b & S_q & R & I_\delta \end{array}\right). \quad (18)$$

The available values for $ker(\Phi(\mathcal{C}))$ depends on the number of columns of S_q, $s = \beta - (\gamma - \kappa) - \delta$.

$$\mathcal{G}' = \left(\begin{array}{cc|ccc} \cancel{I_\kappa} & \cancel{T_b} & \cancel{2T_2} & \cancel{\mathbf{0}} & \cancel{\mathbf{0}} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S_b & S_q & R & I_\delta \end{array} \right).$$

Theorem 40 (BDFT19).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix in standard form, and let $\mathcal{C}'$ be the subcode generated by $\mathcal{G}'$. Then,

$$\ker(\Phi(\mathcal{C})) = \kappa + \ker(\phi(\mathcal{C}'_Y)).$$

Kernel dimension of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

Theorem 41 (FPV10).

Let $\alpha, \beta, \gamma, \delta, \kappa$ be allowable parameters. Then, there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $\ker(C) = \gamma + 2\delta - \bar{k}$ if and only if

$$\begin{cases} \bar{k} = 0, & \text{if} \quad s = 0, \\ \bar{k} \in \{0\} \cup \{2, \ldots, \delta\} \text{ and } \bar{k} \text{ even}, & \text{if} \quad s = 1, \\ \bar{k} \in \{0\} \cup \{2, \ldots, \delta\}, & \text{if} \quad s \geq 2, \end{cases}$$

and $s = \beta - (\gamma - \kappa) - \delta$.

Example 43.

- Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of type $(\alpha, 7; 2, 5; 1)$. Then, $s = 1 \to \bar{k} \in \{0, 2, 4\}$ and $\ker(C) \in \{8, 10, 12\}$.
- Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of type $(\alpha, 8; 2, 5; 1)$. Then, $s = 2 \to \bar{k} \in \{0, 2, 3, 4, 5\}$ and $;\ker(C) \in \{7, 8, 9, 10, 12\}$.

Example

For $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}$ of type $(\alpha, 8; 2, 5; 1)$,
$k = \ker(\Phi(\mathcal{C})) \in \{7, 8, 9, 10, -, 12\}$.

We obtain $\mathbb{Z}_2\mathbb{Z}_4$-linear codes $C = \Phi(\mathcal{C})$ for all possible k, taking:

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} 1 & T_b & \mathbf{0} & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} \\ \hline \mathbf{0} & S_b & S_q & \mathbf{0} & I_5 \end{array}\right)$$

$k = 12$, when $S_q = (\mathbf{0})$ $\quad$ $k = 10$, when $S_q = A$ $\quad$ $k = 9$, when $S_q = B$
$k = 8$, when $S_q = C$ $\quad$ $k = 7$, when $S_q = D$

$$A = \begin{pmatrix} \mathbf{1} & 0 \\ \mathbf{1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & \mathbf{1} \\ 0 & \mathbf{1} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \mathbf{1} & 1 \\ \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & \mathbf{1} \\ 0 & \mathbf{1} \end{pmatrix}$$

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and $I = \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, m\}$. Then, denote

$$\mathbf{v}_I = \mathbf{v}_{i_1} + \cdots + \mathbf{v}_{i_l}.$$

If $I = \emptyset$, then $\mathbf{v}_I = \mathbf{0}$.

Proposition 42 (FPV10).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix $\mathcal{G}$, and let $C = \Phi(\mathcal{C})$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code with $\ker(C) = \gamma + 2\delta - \bar{k}$, where $\bar{k} \in \{2, \ldots, \delta\}$. Let $\{\mathbf{v}_j\}_{j=1}^{\delta}$ be the rows of order four in $\mathcal{G}$. Then, there exists a set $\{j_1, \ldots, j_{\bar{k}}\} \subseteq \{1, \ldots, \delta\}$ such that

$$C = \bigcup_{I \subseteq \{j_1, \ldots, j_{\bar{k}}\}} (K(C) + \Phi(\mathbf{v}_I)).$$

Example 44.

Let $\mathcal{C}_{15}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive defined before. We have that

$$\mathcal{C}_{15} = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle,$$
$$\mathcal{K}(\mathcal{C}_{15}) = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, 2\mathbf{v}_2, 2\mathbf{v}_3 \rangle.$$

We can write $C_{15} = \Phi(\mathcal{C}_{15})$ as the following union of cosets of $K(C_{15})$:

$$\begin{aligned} C_{15} = & K(C_{15}) \cup \\ & \big(K(C_{15}) + \Phi(\mathbf{v}_2)\big) \cup \\ & \big(K(C_{15}) + \Phi(\mathbf{v}_3)\big) \cup \\ & \big(K(C_{15}) + \Phi(\mathbf{v}_2 + \mathbf{v}_3)\big). \end{aligned}$$

4 Linearity, Rank and Kernel

- Basic definitions
- Linearity
- Rank of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes
- Kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes
- Pairs of rank and dimension of the kernel

Proposition 43 (FPV10).

Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $\ker(C) = \gamma + 2\delta - \bar{k}$ and $\mathrm{rank}(C) = \gamma + 2\delta + \bar{r}$. Then, for any $\bar{k} \in \{0\} \cup \{2, \ldots, \delta\}$,

$$\begin{cases} \bar{r} = 0, & \text{if } \bar{k} = 0, \\ \bar{r} \in \{2, \ldots, \min\{\beta - (\gamma - \kappa) - \delta, \binom{\bar{k}}{2}\}\}, & \text{if } \bar{k} \text{ is odd}, \\ \bar{r} \in \{1, \ldots, \min\{\beta - (\gamma - \kappa) - \delta, \binom{\bar{k}}{2}\}\}, & \text{if } \bar{k} > 0 \text{ is even}. \end{cases}$$

Existence

Theorem 44 (FPV10).

Let $\alpha, \beta, \gamma, \delta, \kappa$ be allowable parameters. Then, there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $\ker(C) = \gamma + 2\delta - \bar{k}$ and $\operatorname{rank}(C) = \gamma + 2\delta + \bar{r}$ if and only if $\bar{k} \in \{0\} \cup \{2, \ldots, \delta\}$ and

$$\begin{cases} \bar{r} = 0, & \text{if } \bar{k} = 0, \\ \bar{r} \in \{2, \ldots, \min\{\beta - (\gamma - \kappa) - \delta, \binom{\bar{k}}{2}\}\}, & \text{if } \bar{k} \text{ is odd}, \\ \bar{r} \in \{1, \ldots, \min\{\beta - (\gamma - \kappa) - \delta, \binom{\bar{k}}{2}\}\}, & \text{if } \bar{k} > 0 \text{ is even}. \end{cases}$$

Example 45.

The possible pairs of rank and dimension of the kernel, (r, k) for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of type $(\alpha, 9; 2, 5; 1)$, are the ones given in the following table:

$k \setminus r$	12	13	14	15
12	*			
10		*		
9			*	*
8		*	*	*
7			*	*

Example 46 (cont.).

For each possible pair (r,k), we can construct a $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C_{r,k}$ with $\mathrm{rank}(C_{r,k}) = r$ and $\ker(C_{r,k}) = k$, taking the following generator matrix of $\mathcal{C}_{r,k} = \Phi^{-1}(C_{r,k})$:

$$\mathcal{G}_{r,k} = \left(\begin{array}{cc|ccc} 1 & T_b & \mathbf{0} & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} \\ \hline \mathbf{0} & S_b & S_{r,k} & \mathbf{0} & I_5 \end{array}\right),$$

where T_b, S_b are matrices over $\mathbb{Z}_2$; and the matrices $S_{r,k}$, for each $(r,k) \in \{(12,12),(13,10),(13,8),(14,9),(14,8),(14,7),$ $(15,9),(15,8),(15,7)\}$, are the following: $S_{12,12} = (\mathbf{0})$,

Example 47 (cont.).

$$\mathcal{G}_{r,k} = \left(\begin{array}{cc|ccc} 1 & T_b & \mathbf{0} & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} \\ \hline \mathbf{0} & S_b & S_{r,k} & \mathbf{0} & I_5 \end{array}\right)$$

$$S_{13,10} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{13,8} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Example 48 (cont.).

$$S_{14,9} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{14,8} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{14,7} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$S_{15,9} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{15,8} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{15,7} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

MAGMA. Computational Algebra System

`http://magma.maths.usyd.edu.au/magma/`
`http://www.ccsg.uab.cat` (Downloads/Z2Z4-Additive Codes version 4.0)

Some functions for linearity, rank and kernel of $\mathbb{Z}_2\mathbb{Z}_4$-**additive codes**:

- `HasZ2Z4LinearGrayMapImage(C)`
- `Z2Z4SpanZ2Code(C)`
- `Z2Z4KernelZ2Code(C)`
- `Z2Z4KernelCosetRepresentatives(C)`
- `Z2Z4DimensionOfSpanZ2(C)`
- `Z2Z4RankZ2(C)`
- `Z2Z4DimensionOfKernelZ2(C)`

5 ACD codes

- Basic definitions and characterization
- Complemantary duality of $\mathcal{C}$, $\mathcal{C}_X$ and $\mathcal{C}_Y$.
- Binary LCD codes from ACD codes

Bibliography

[BBD+20] N. Benbelkacem, J. Borges, S.T. Dougherty, C. Fernández-Córdoba.
On $\mathbb{Z}_2\mathbb{Z}_4$-additive complementary dual codes and related LCD codes
Finite Fields Appl., vol. 62, 2020.

5 ACD codes

- Basic definitions and characterization
- Complemantary duality of $\mathcal{C}$, $\mathcal{C}_X$ and $\mathcal{C}_Y$.
- Binary LCD codes from ACD codes

LCD and ACD codes

A binary (or quaternary) code C is said to be *linear complementary dual* (LCD) if it is linear and $C \cap C^\perp = \{\mathbf{0}\}$ [Mas92].

A code $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is *additive complementary dual* (briefly ACD) if it is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code such that $\mathcal{C} \cap \mathcal{C}^\perp = \{\mathbf{0}\}$ [BBD+20].

[Mas92] J.L. Massey.
Linear Codes with Complementary Duals
Disc. Math, 106/107, pp. 337-342, 1992.

LCD and ACD codes

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[Mas92] J.L. Massey.
Linear Codes with Complementary Duals
Disc. Math, 106/107, pp. 337-342, 1992.

What may be interesting on ACD codes?

- Characterization of ACD codes.
- Relationship between complemantary duality of $\mathcal{C}$, $\mathcal{C}_X$ and $\mathcal{C}_Y$.
- Relationship between complemantary duality of $\mathcal{C}$ and $\Phi(\mathcal{C})$.

Characterization of ACD codes.

Lemma 49 (Mas92).

Let C be a binary LCD code. Then $\mathbb{Z}_2^n = C \oplus C^\perp$. That is, any vector w in $\mathbb{Z}_2^n$ can be written uniquely as $w_1 + w_2$, for $w_1 \in C$ and $w_2 \in C^\perp$.

Proposition 45 (Mas92).

Let C be a binary (n, k) linear code with generator matrix G and parity-check matrix H. The following statements are equivalent:

1. *C is an LCD code,*
2. *the $k \times k$ matrix GG^T is nonsingular,*
3. *the $(n-k) \times (n-k)$ matrix HH^T is nonsingular.*

Characterization of ACD codes.

Lemma 50 (BBD+20).

Let $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ be an ACD code. Then any vector $\mathbf{w} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ can be written uniquely as $\mathbf{w}_1 + \mathbf{w}_2$, for $\mathbf{w}_1 \in \mathcal{C}$ and $\mathbf{w}_2 \in \mathcal{C}^\perp$.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix $\mathcal{G} = (G_X \mid G_Y)$. We define the product

$$\mathcal{G} \cdot \mathcal{G}^t = \left(G_X \mid G_Y \right) \cdot \left(\frac{G_X^t}{G_Y^t} \right) = 2\iota(G_X)\iota(G_X)^t + G_Y G_Y^t.$$

Characterization of ACD codes.

Lemma 50 (BBD+20).

Let $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ be an ACD code. Then any vector $\mathbf{w} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ can be written uniquely as $\mathbf{w}_1 + \mathbf{w}_2$, for $\mathbf{w}_1 \in \mathcal{C}$ and $\mathbf{w}_2 \in \mathcal{C}^\perp$.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix $\mathcal{G} = (G_X \mid G_Y)$. We define the product

$$\mathcal{G} \cdot \mathcal{G}^t = \left(\; G_X \mid G_Y \; \right) \cdot \left(\frac{G_X^t}{G_Y^t} \right) = 2\iota(G_X)\iota(G_X)^t + G_Y G_Y^t.$$

Characterization of ACD codes.

Proposition 46 (BBD+20).

Let G be a generator matrix for a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ and consider the matrix $G \cdot G^T = (w_{ij})_{1 \leq i,j \leq r}$ with entries from $\mathbb{Z}_4$. If $w_{ij} \in \{0,2\}$ and $w_{ii} \notin \{0,2\}$ for all $i,j = 1,\ldots,r$ such that $i \neq j$, then $\mathcal{C}$ is an ACD code and $\mathcal{C}_Y$ is a quaternary LCD code.

The reverse is not true in general.

Characterization of ACD codes.

Proposition 46 (BBD+20).

Let G be a generator matrix for a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ and consider the matrix $G \cdot G^T = (w_{ij})_{1\leq i,j \leq r}$ with entries from $\mathbb{Z}_4$. If $w_{ij} \in \{0,2\}$ and $w_{ii} \notin \{0,2\}$ for all $i,j = 1,\ldots,r$ such that $i \neq j$, then $\mathcal{C}$ is an ACD code and $\mathcal{C}_Y$ is a quaternary LCD code.

The reverse is not true in general.

5 ACD codes

- Basic definitions and characterization
- Complemantary duality of $\mathcal{C}$, $\mathcal{C}_X$ and $\mathcal{C}_Y$.
- Binary LCD codes from ACD codes

Proposition 47 (BBD+20).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. If $\mathcal{C}$ is separable, then $\mathcal{C}$ is an ACD code if and only if $\mathcal{C}_X$ is a binary LCD code and $\mathcal{C}_Y$ is a quaternary LCD code.

What happens if $\mathcal{C}$ is not separable?

Proposition 47 (BBD+20).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. If $\mathcal{C}$ is separable, then $\mathcal{C}$ is an ACD code if and only if $\mathcal{C}_X$ is a binary LCD code and $\mathcal{C}_Y$ is a quaternary LCD code.

What happens if $\mathcal{C}$ is not separable?

$\mathcal{C}_X$ and $\mathcal{C}_Y$ LCD

Example 51.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by

$$\left(\begin{array}{c|c} I_\alpha & I_\alpha \\ \mathbf{1} & \mathbf{2} \end{array} \right).$$

- $\mathcal{C}_X = \mathbb{Z}_2^\alpha$ is an LCD code.
- $\mathcal{C}_Y = \mathbb{Z}_4^\alpha$ is also LCD.
- $(\mathbf{1} \mid \mathbf{2}) \in \mathcal{C} \cap \mathcal{C}^\perp$ and $\mathcal{C}$ is not ACD.

Non-separable ACD codes

Given a non-separable ACD code $\mathcal{C}$ there are examples of all possible situations:

- Both $\mathcal{C}_X$ and $\mathcal{C}_Y$ are LCD codes.
- Both $\mathcal{C}_X$ and $\mathcal{C}_Y$ are not LCD codes.
- $\mathcal{C}_X$ is a LCD code and $\mathcal{C}_Y$ is not.
- $\mathcal{C}_Y$ is a LCD code and $\mathcal{C}_X$ is not.

$\mathcal{C}$ ACD, $\mathcal{C}_X$, $\mathcal{C}_Y$ LCD

Example 52.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by

$$\mathcal{G} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{array} \right).$$

$$\mathcal{G} \cdot \mathcal{G}^t = \left(\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{array} \right).$$

Therefore, $\mathcal{C}$ is ACD. Moreover, $\mathcal{C}_X$ and $\mathcal{C}_Y$ are both LCD codes.

$\mathcal{C}$ ACD and neither $\mathcal{C}_X$ nor $\mathcal{C}_Y$ LCD

Example 53.

Let $\mathcal{C}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-code with generator matrix, and parity check matrix

$$\mathcal{G} = \left(\begin{array}{ccc|cc} 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right), \quad \mathcal{H} = \left(\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right),$$

respectively.

- $\mathcal{C}$ is an ACD code since $\mathcal{C} \cap \mathcal{C}^\perp = \{\mathbf{0}\}$.
- $(1,1,0) \in \mathcal{C}_X \cap \mathcal{C}_X^\perp$.
- $(2,0) \in \mathcal{C}_Y \cap \mathcal{C}_Y^\perp$.

$\mathcal{C}$ ACD and either $\mathcal{C}_X$ or $\mathcal{C}_Y$ LCD

Example 54.

Let D_1 be a binary (α, δ) self-orthogonal code with generator matrix G_X. Let $\mathcal{C}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive code generatorated by

$$\mathcal{G} = (G_X \mid I_\delta).$$

- $\mathcal{C}_X$ is self-orthogonal and hence not LCD.
- $\mathcal{C}_Y = \mathbb{Z}_4^\alpha$ is LCD.
- $\mathcal{C}$ is ACD.

$\mathcal{C}$ ACD and either $\mathcal{C}_X$ or $\mathcal{C}_Y$ LCD

Example 55.

Let $\mathcal{C}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additiv code generated by

$$\mathcal{G} = (I_\alpha \mid 2I_\alpha)\,.$$

Then, $\mathcal{C}_X$ is a binary LCD code and $\mathcal{C}_Y$ is not a quaternary LCD code because it is a self-dual code.

- $\mathcal{C}_X$ is a binary LCD code.
- $\mathcal{C}_Y$ is self-dual and hence not LCD.
- $\mathcal{C}$ is ACD.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. When is $C = \Phi(\mathcal{C})$ an LCD code?

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \Phi\ } & C = \Phi(\mathcal{C}) \\ {\scriptstyle\perp}\big\downarrow & & \\ \mathcal{C}^{\perp} & \xrightarrow{\ \Phi\ } & C_{\perp} = \Phi(\mathcal{C}^{\perp}) \end{array}$$

- Maybe the diagram does not commute.
- Maybe C is not linear.
- Maybe $C_{\perp}$ is not linear.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. When is $\mathcal{C} = \Phi(\mathcal{C})$ an LCD code?

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Phi} & C = \Phi(\mathcal{C}) \\ \perp\downarrow & & \\ \mathcal{C}^\perp & \xrightarrow{\Phi} & C_\perp = \Phi(\mathcal{C}^\perp) \end{array}$$

- Maybe the diagram does not commute.
- Maybe C is not linear.
- Maybe $\mathcal{C}_\perp$ is not linear.

Theorem 56 (BBD+20).

Let $\mathcal{C}$ be an ACD code, $C = \Phi(\mathcal{C})$, $C_\perp = \Phi(\mathcal{C}^\perp)$ and

$$D_{\mathcal{C}} = \{2\mathbf{u} * \mathbf{v} \mid \mathbf{u} \in \mathcal{C}, \mathbf{v} \in \mathcal{C}^\perp\}.$$

The following statements are equivalents:

(i) C *is linear and* $D_{\mathcal{C}} \subseteq \mathcal{C}$.

(ii) $C_\perp$ *is linear and* $D_{\mathcal{C}} \subseteq \mathcal{C}^\perp$.

(iii) C *and* $C_\perp$ *are linear.*

(iv) $D_{\mathcal{C}} = \{\mathbf{0}\}$.

(v) C *and* $C_\perp$ *are LCD.*

(vi) $C_\perp = C^\perp$.

Bibliography

[BBDF11] M. Bilal, J. Borges, S. T. Dougherty, C Fernández-Córdoba.
Maximum distance separable codes over $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$
Designs, Codes and Cryptography, vol. 61, pp. 31-40, 2011.

Hamming,Lee distance

The *Hamming distance* $d_H(u,v)$ between two vectors $u=(u_1,\ldots,u_n), v=(v_1\ldots,v_n)\in\mathbb{Z}_2^n$ is

$$d_H(u,v)=|\{i\in\{1,\ldots,n\}:u_i\neq v_i\}|$$

The *minimum Hamming distance* $d_H(C)$ of a binary code C is

$$d_H(C)=\min\{d_H(u,v):u,v\in C,u\neq v\}.$$

Lee distance

The *Lee weights* over the elements in $\mathbb{Z}_4$ are defined as $\mathrm{wt}_L(0) = 0$, $\mathrm{wt}_L(1) = \mathrm{wt}_L(3) = 1$, and $\mathrm{wt}_L(2) = 2$. Then, the *Lee weight* of a vector $u = (u_1 \ldots, u_n) \in \mathbb{Z}_4^n$ is

$$\mathrm{wt}_L(u) = \sum_{i=1}^{n} \mathrm{wt}_L(u_i).$$

The *Lee distance* $d_L(u, v)$ between two vectors $u, v \in \mathbb{Z}_4^n$ is

$$d_L(u, v) = \mathrm{wt}_L(u - v).$$

The *minimum Lee distance* $d_L(\mathcal{C})$ of a quaternary code $\mathcal{C}$ is

$$d_L(\mathcal{C}) = \min\{d_L(u, v) : u, v \in \mathcal{C}, u \neq v\},$$

Given two elements $\mathbf{u} = (u \mid u'), \mathbf{v} = (v \mid v') \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, we define the distance between $\mathbf{u}$ and $\mathbf{v}$ as

$$d(\mathbf{u}, \mathbf{v}) = d_H(u, v) + d_L(u', v').$$

The minimum distance of $\mathcal{C}$ is defined as

$$d(\mathcal{C}) = \min\{d(\mathbf{u}, \mathbf{v}) : \mathbf{u}, \mathbf{v} \in \mathcal{C} \text{ and } \mathbf{u} \neq \mathbf{v}\}.$$

It is easy to see that

$$d(\mathbf{u}, \mathbf{v}) = d_H(\Phi(\mathbf{u}), \Phi(\mathbf{v})),$$

$$d(\mathcal{C}) = d_H(\Phi(\mathcal{C})).$$

Given two elements $\mathbf{u} = (u \mid u'), \mathbf{v} = (v \mid v') \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, we define the distance between $\mathbf{u}$ and $\mathbf{v}$ as

$$d(\mathbf{u}, \mathbf{v}) = d_H(u, v) + d_L(u', v').$$

The minimum distance of $\mathcal{C}$ is defined as

$$d(\mathcal{C}) = \min\{d(\mathbf{u}, \mathbf{v}) : \mathbf{u}, \mathbf{v} \in \mathcal{C} \text{ and } \mathbf{u} \neq \mathbf{v}\}.$$

It is easy to see that

$$d(\mathbf{u}, \mathbf{v}) = d_H(\Phi(\mathbf{u}), \Phi(\mathbf{v})),$$

$$d(\mathcal{C}) = d_H(\Phi(\mathcal{C})).$$

Singleton bound for binary codes

Let C be a binary code of length n and dimension K. The usual Singleton bound for C [Sing64] is

$$d_H(C) \leq n - \log_2 |C| + 1 = n - k + 1.$$

The only binary codes achieving this bound are repetition codes and universe codes [MS77].

[Sing64] R. Singleton.
Maximum distance q-nary codes
IEEE Transactions on Information Theory, vol. 10, pp. 116-118, 1964.

[MS77] F. J. MacWilliams, N. J. A. Sloane.
The Theory of Error-correcting Codes
Elsevier, 1977.

Singleton bound for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $C = \Phi(\mathcal{C})$. Since $d(\mathcal{C}) = d(C)$, we have [BBDF11]

$$d(\mathcal{C}) \leq \alpha + 2\beta - \gamma - 2\delta + 1. \quad (19)$$

(For quaternary linear codes in [DS01])

[DS01] S. T. Dougherty, K. Shiromoto.
Maximum distance codes over rings of order 4
IEEE Transactions on Information Theory, vol. 47, pp. 400-404, 2001.

Rank related bound

From [DS01], if $\mathcal{C}$ is a code of length n over a ring R with minimum distance $d(\mathcal{C})$, then

$$\left\lfloor \frac{d(\mathcal{C})-1}{2} \right\rfloor \leqslant n - \mathrm{rank}(\mathcal{C}), \tag{20}$$

where $rank(\mathcal{C})$ is the minimal cardinality of a generating system for $\mathcal{C}$.

Theorem 48 (BBDF11).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then,

$$\frac{d(\mathcal{C}) - 1}{2} \leqslant \frac{\alpha}{2} + \beta - \frac{\gamma}{2} - \delta; \tag{21}$$

$$\left\lfloor \frac{d(\mathcal{C}) - 1}{2} \right\rfloor \leqslant \alpha + \beta - \gamma - \delta. \tag{22}$$

- Singleton bound $\longrightarrow$ (21)
- Rank related bound $\longrightarrow$ (22)

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. We say that $\mathcal{C}$ is

- *maximum distance separable* (MDS) if $d(\mathcal{C})$ meets the bound given in (21) or (22).
- MDS with respect to the Singleton bound (MDSS) if it meets bound given in (21).
- MDS with respect to the rank bound (MDSR) if it meets bound given in (22).

6 Maximum Distance Separable $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

- Basic definitions
- Characterization of MDS $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

MDSS codes

Theorem 49 (BBDF11).

Let $\mathcal{C}$ be an MDSS $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ such that $1 < |\mathcal{C}| < 2^{\alpha+2\beta}$. Then, $\mathcal{C}$ is either

(i) *the repetition code of type $(\alpha, \beta; 1, 0; \kappa)$ and minimum distance $d(\mathcal{C}) = \alpha + 2\beta$, where $\kappa = 1$ if $\alpha > 0$ and $\kappa = 0$ otherwise; or*

(ii) *the even code with minimum distance $d(\mathcal{C}) = 2$ and type $(\alpha, \beta; \alpha - 1, \beta; \alpha - 1)$ if $\alpha > 0$, or type $(0, \beta; 1, \beta - 1; 0)$ otherwise.*

Note that the codes described in (i) and (ii) of last theorem $\mathbb{Z}_2\mathbb{Z}_4$-additive dual codes. Hence, the $\mathbb{Z}_2\mathbb{Z}_4$-additive dual of a MDS code is also an MDS code.

This property is well known property for linear codes over finite fields [MS77].

Note that the codes described in (i) and (ii) of last theorem $\mathbb{Z}_2\mathbb{Z}_4$-additive dual codes. Hence, the $\mathbb{Z}_2\mathbb{Z}_4$-additive dual of a MDS code is also an MDS code.

This property is well known property for linear codes over finite fields [MS77].

MDSR codes

Theorem 50 (BBDF11).

Let $\mathcal{C}$ be an MDSR $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ such that $1 < |\mathcal{C}| < 2^{\alpha+2\beta}$. Then, either

(i) *$\mathcal{C}$ is the repetition code as in with $\alpha \leq 1$; or*

(ii) *$\mathcal{C}$ is of type $(\alpha, \beta; \gamma, \alpha+\beta-\gamma-1; \alpha)$, where $\alpha \leq 1$ and $d(\mathcal{C}) = 4 - \alpha \in \{3, 4\}$; or*

(iii) *C is of type $(\alpha, \beta; \gamma, \alpha+\beta-\gamma; \alpha)$, where $\alpha \leq 1$ and $d(\mathcal{C}) \leq 2 - \alpha \in \{1, 2\}$.*

Example 57.

Let $\mathcal{C}_2$ be the $(1,1;0,1;0)$ $\mathbb{Z}_2\mathbb{Z}_4$-additive code of length 2 with generator matrix

$$\mathcal{G}_2 = (1|1).$$

We have $d(\mathcal{C}_2) = 2$ and

$$\frac{d(\mathcal{C}_2)-1}{2} = \frac{\alpha}{2} + \beta - \frac{\gamma}{2} - \delta;$$

$$\left\lfloor \frac{d(\mathcal{C}_2)-1}{2} \right\rfloor < \alpha + \beta - \gamma - \delta.$$

and it is a MDSS code (it is the even code of lenght 3) and not an MDSR code.

Example 58 (cont.).

Its $\mathbb{Z}_2\mathbb{Z}_4$-additive dual code $\mathcal{C}_2^{\perp}$ is the repetition code

$$\{(0,0),(1,2)\}$$

of type $(\bar{\alpha},\bar{\beta};\bar{\gamma},\bar{\delta};\bar{\kappa}) = (1,1;1,0;1)$. Note that

$$\frac{d(\mathcal{C}_2^{\perp})-1}{2} = \frac{\bar{\alpha}}{2} + \bar{\beta} - \frac{\bar{\gamma}}{2} - \bar{\delta};$$

$$\left\lfloor \frac{d(\mathcal{C}_2^{\perp})-1}{2} \right\rfloor = \bar{\alpha} + \bar{\beta} - \bar{\gamma} - \bar{\delta}.$$

Then, $\mathcal{C}_2^{\perp}$ is MDSS and MDSR.